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Modeling Product Diffusion and Repeat Purchases: A Fractional Calculus-Based Approach

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Abstract

Classic product diffusion models such as the Bass Model typically consider only initial product purchases. For products that experience frequent upgrades or have multiple versions, repeat product purchases can constitute a significant proportion of sales. Despite the long tradition of product diffusion research, there exists no viable model option when repeat purchases are inseparable from initial ones in sales data. The present study proposes a new sales growth model, termed the *Generalized Diffusion Model with Repeat Purchases* (GDMR), to fill this void. The GDMR treats the sales process as *an economic process with memory* and formulates the growth rate of sales using a non-integer-order integral equation rather than an integer-order differential equation typically used in existing diffusion models. The GDMR is parsimonious and easy to implement. Empirical results show that the GDMR fits sales data with varying proportions of repeat purchases, thus making it a suitable model for a wide variety of products. We also demonstrate that the GDMR can effectively recover adoption (i.e., initial sales) curves when only sales data is available, thus underscoring its theoretical validity. Furthermore, the GDMR can be extended to incorporate marketing mix variables, further enhancing its value in managerial decision-making.

Keywords: Diffusion of innovation, repeat purchases, replacement, multi-unit ownerships, fractional calculus

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1. Introduction

Pioneering diffusion of innovation studies have suggested that the cumulative diffusion of an innovation follows an S-shaped curve and the noncumulative diffusion follows a bell-shaped curve (Rogers 2003). Much of the effort in modeling the diffusion of innovation has been dedicated to developing mathematical formulations that possess such properties. The seminal Bass Model (Bass 1969) is probably the most well-known result from such endeavors. Since its inception, thanks to its great empirical performance and ease of implementation, the Bass Model has spawned a large body of literature covering diverse fields such as marketing, operations, information systems, and other nonbusiness disciplines. Researchers have extended and applied the Bass Model to study the diffusion of products including durable goods, nondurable goods, information products, and business practices such as IT outsourcing (Bass, Krishnan, and Jain 1994, Hu, Saunders, and Gebelt 1997, Mahajan, Muller, and Wind 2000, Bass 2004).

As a diffusion model, the Bass Model is concerned "only with the timing of the initial purchases" (Bass 1969). If only initial purchases (or simply, *adoptions*) are counted, the market saturation effect comes into play; hence, after a market peak is reached, the rate of adoptions drops monotonically and asymptotically approaches zero. This trend is illustrated by the symmetric dashed curve shown in Figure 1.

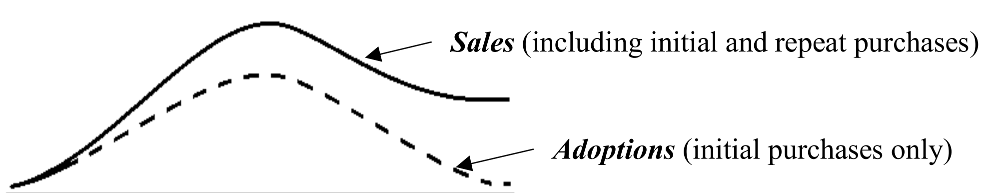


Figure 1. Sales vs. Adoptions

In practice, however, only adoption data is rarely available. What is typically available is *sales data* that includes both *initial* and *repeat* purchases. The plot of sales data often exhibits an asymmetrical bell-shaped pattern, as illustrated by the solid curve shown in Figure 1. In fact, Bass (1969) himself points out that "sales often grow to a peak and then level off at some magnitude lower than the peak" and that the "stabilizing effect is accounted for by the relative growth of the replacement purchasing component of sales and the decline of the initial purchase component" (Bass 1969, p. 215).

Even though Bass acknowledges that his model is concerned with initial purchases only, the Bass Model is frequently applied to fit sales data that counts both initial purchases and repeat purchases. Such applications are reasonable when repeat purchases are infrequent or negligible, which might be the case for durable products during the early stages of product sales.

For products that have frequent upgrades or multiple versions, repeat purchases can constitute a significant proportion of sales even in the early stages of product diffusion, thus limiting the application of the Bass Model. When significant repeat purchases exist, the fitting and forecasting performance of the Bass Model may be poor, particularly when the sales trend begins to assume an asymmetrical shape after the peak point of diffusion. Even if the model fits and predicts well in some situations despite the existence of considerable repeat purchases, the parameter estimates will be different from those obtained using only initial purchases data.

Furthermore, existing adopters' enthusiasm toward an existing product tends to die down with time, and products built on newer technologies will inevitably emerge, hence existing adopters' tendencies to make repeat purchases tend to decline over time. This creates an additional challenge in accurately modeling the rate of repeat purchases and subsequently sales.

The primary objective of this study is to develop a generalized sales growth model that incorporates both initial and repeat purchases and performs well despite the challenges.

In the prior literature, researchers have proposed model extensions to account for repeat product purchases made to replace existing product units or for adopting multiple product units. Olson and Choi (1985) propose a model assuming that sales comprise only adoptions and replacements, and that the replacement hazard function follows the Rayleigh distribution (Papoulis and Pillai 2002). The Olson-Choi Model is developed for cases in which the number of products in use is available.

Kamakura and Balasubramanian (1987) propose a similar model that generates long-term forecasts by incorporating the adoption and replacement components of sales. Benefiting from a more flexible hazard function to capture replacement purchases, their model is applicable with or without data for replacement sales. Their model uses information from similar products when data for replacement purchases is not available. Steffens and Balasubramanian (1998) further advance the modeling of replacement sales by allowing the distribution of the service life of replaced products to vary over time.

Focusing on the PC processor industry, Gordon (2009) introduces a dynamic structural model that explicitly considers product replacement decisions under uncertain future product quality and price. Gordon's model concentrates specifically on product replacements due to obsolescence as a result of product upgrade releases. The model requires highly specific data to function as it uses a composite dataset including sales, ownership, price, and product quality. Due to data limitations, Gordon's model does not consider multi-unit ownership purchases (e.g., three TV sets in a household).

The aforementioned models have their focus on adoptions and replacement purchases. There are other models from the prior literature that have considered other components of sales, in particular multi-unit ownership purchases, in addition to adoptions and replacements. Dodson and Muller (1978) propose a model that captures the type of asymmetric sales trend illustrated by the solid curve in Figure 1 without decomposing sales into multiple components. The market they consider is composed of three groups, i.e., those who are not aware of the product, those who are aware of the product but have not made a purchase, and those who have purchased the product. Although the portrayed interaction between adopters and non-adopters is insightful, their model cannot be operationalized if the data for the different market groups is not available.

Based on econometric and simulation models, Bayus, Hong, and Labe (1989) incorporate first time sales, replacement sales, additional-unit sales, and institutional sales into their analysis of color television set sales. As stated by Steffens (2003), Bayus et al.'s model is developed to perform well over short terms. Steffens (2003) presents a model for sales resulting from multi-unit ownership purchases based on the Bass Model. His model differs from Bayus et al.'s in that it imposes a saturation level on multi-unit ownerships, which makes the model applicable for longer time frames.

Researchers have also developed models for repeat purchases of products for a specific industry. For instance, Lilien, Rao, and Kalish (1981) propose a highly specialized model to project sales of prescription drugs as a function of a focal pharmaceutical company's own detailing effect, competitors' detailing effect, and word of mouth. In a subsequent study, Rao and Yamada (1988) provide further empirical support for the model by Lilien, Rao, and Kalish, then propose an alternative method for developing priors for the new drug's parameters, and demonstrate how the parameters can be updated after sales data becomes available. Because

Lilien, Rao, and Kalish's and Rao and Yamada's models are specifically designed for prescription drugs, they cannot be used to predict sales of other product categories.

Our review of the literature suggests that there is a need for a comprehensive model that can fit aggregate sales data that records both initial purchases and repeat purchases including both replacements and multi-unit ownerships. The models presented in the extant literature are applicable mostly when separate data for distinct sales components are available; however, in practice, most of the times only aggregate product sales data is available. Therefore, in this study, we develop a model that can capture the sales pattern illustrated by the solid curve in Figure 1 without separating different sales components.

The model we propose is termed as the *Generalized Diffusion Model with Repeat Purchases* (GDMR). By treating the sales process as an *economic process with memory*, we are able to utilize a branch of mathematics named *fractional calculus* to develop this model. Specifically, the GDMR generalizes the fundamental differential equation governing the Bass Model, and employs a non-integer integral operator with flexible order, thereby rendering the Bass Model a special case of the extended model. The GDMR adopts an approach that is different from those in the prior literature in that it captures the sales growth rate using a non-integer order integral equation, rather than an integer-order differential equation as used in the prior literature.

Compared with the existing models, the GDMR has the following important advantages:

- Developed by adding only one parameter to the classic Bass Model, the GDMR retains the Bass Model's parsimony as well as its insightful behavioral explanation concerning adopters' decisions in a diffusion process, which have been two of the main reasons for its extensive application in academic research and practice.

- The GDMR can count both replacement and multi-unit ownership purchases, further broadening its application compared to models that include only one of these repeat-purchase components. Further, the GDMR is easy to implement and can fit sales data with varying frequencies of repeat purchases, making it a suitable model for a wide variety of sales scenarios.
- Marketing mix variables can be incorporated into the GDMR, thus further enhancing its potential in helping firms make better marketing decisions.

In addition to these theoretical advantages, our empirical analysis shows that the GDMR delivers superior performance in both model fitting and forecasting relative to other models proposed in the extant literature.

The rest of this paper is organized as follows. In Section 2, we discuss what constitutes repeat purchases. Section 3 presents the model development of the GDMR. Empirical testing and related discussions are summarized in Section 4. Concluding remarks are presented in Section 5.

2. Defining Repeat Purchases

Before delving into model development, we first discuss the drivers of repeat purchases and a conceptual framework that helps explain the types of innovations and clarify what qualify as repeat purchases. In other words, we explain our unit of analysis that helps define the scope of adoptions and repeat purchases.

Repeat purchases may result from either product *replacements* (e.g., buying a new TV set to replace an old one) or *multi-unit ownerships* (e.g., one household owning multiple TV sets). Drivers of replacement purchases vary from nondurable to durable products. For nondurable products, replacement usually occurs as the result of consumption (Kamakura and Balasubramanian 1987). For durable products, replacement typically takes place when a product under consumption fails to meet the requirements of the user (Kamakura and Balasubramanian 1987; Steffens 2003). This failure may be caused by perceived or actual wear and tear as a result of consumption or by changes in a user's own needs or expectations that can only be met by a different edition or generation of the original product. In high-tech markets, in particular, replacement is often driven as much by product obsolescence as wear and tear, as these markets experience frequent changes in the forms of quality improvements and/or price decreases (Gordon 2009).

Motivations behind multi-unit adoptions may vary (Steffens 2003). Common reasons include, for example, using the product in different locations (e.g., TV sets for different rooms of a house, desktop computers for office and for home), providing extra capacity for peak demand (e.g., multiple photocopiers at the same location), or meeting the demand for different functionalities that require multiple editions of a product (e.g., a sedan vehicle for daily commutes and a sport utility vehicle for leisure trips).

We next draw on the typology of product innovation by Henderson and Clark (1990) to define our unit of analysis — a family of products for which adoptions and repeat purchases take place. Henderson and Clark (1990) classify product innovations based on the amount of corresponding changes to the product's core components and product architecture. For example, a product innovation is considered *radical* if it introduces a new set of designs to the core components as well as a new architecture that links the core components. If the design of the core components and the architecture undergo only small improvements, the innovation is deemed *incremental*. In this study, we count a new purchase as a repeat purchase only if the newly purchased product is the same as the original product or a variation of the original product resulting from an incremental innovation. In other words, our unit of analysis is a line of products/innovations that differ incrementally. Within such a unit of analysis, the first purchase made for any of the products in the product line is considered the initial purchase, substituting a product by another in the same product line constitutes replacement, and simultaneous ownership of multiple units of the product line represents multi-unit ownership.

As technology advances and different products emerge based on innovation types other than incremental innovation, adopters may leave a focal product line and buy new products. When they buy such new products, even if they have similar functionalities, the purchases can no longer be counted as repeat purchases; instead, they are *departures* from the focal product line. For example, we can count the purchases of new models of DVD players as repeat purchases, but we need to treat upgrades to Blu-ray players as departures. While replacements and multi-unit ownerships increase repeat purchases, departures decrease repeat purchases. The goal of this study is to develop a *unified* model that can capture adoptions, replacements, multi-unit ownership purchases, as well as the declining rate of repeat purchase due to departures.

3. Generalized Diffusion Model with Repeat Purchases

Given that product sales typically consist of both adoptions (initial purchases) and repeat purchases, we need to consider the two sales components separately. While the rate of adoptions at any given time t can be estimated using existing diffusion models such as the Bass Model (1969), it is far less straightforward to model the rate of repeat purchases at time t. For example, because existing adopters who have purchased a focal product at different times in the past are not expected to have the same probability of making repeat purchases at the present time, we cannot multiply the number of existing adopters by a constant multiplier to generate the rate of repeat purchases. Instead, to have a reliable estimate of the rate of repeat purchases at time t, a model should not just check the total number of adopters at a given time t, but also *memorize* the focal product's entire adoption growth history by time t.

3.1. Modeling Framework for Repeat Purchases: Economic Process with Memory

After extensive literature review, we find that sales processes with repeat purchases fit the description of *processes with memory* (Boltzmann 1876), or more specifically *economic processes with memory* (Beran 1994, Baillie 1996, Teyssière and Kirman 2006, Palma 2007). According to Tarasov (2018), in an economic process with memory, memory captures the dependence of an output (response variable) at the present time on the history of the changes of an input (impact variable) in a given time frame. Applying Tarasov's memory concept, we can interpret *the sales of a product as an economic process with memory, in which* ***adoptions*** *is the* ***input****,* ***sales*** *is the* ***output****, and* ***memory*** *defines the percentage of adoptions from each time in the* ***past*** *that generate* ***repeat purchases*** *at the* ***present time***.

Regarding the methods used to study economic processes with memory, Tarasov (2018) points out "it is known that derivatives of positive integer orders are determined by the properties of the differentiable function only in an infinitesimal neighborhood of the considered point. As a result, differential equations with integer-order derivatives cannot describe processes with memory." Tarasov then suggests a powerful tool that can be used to describe economic processes with memory. The tool is called *fractional calculus*, which represents a branch of mathematics that generalizes differentiation and integration so that non-integer-order differential and integral operators become possible. Interested readers can refer to Samko, Kilbas, and Marichev (1993), Podlubny (1999), Kilbas, Srivastave, and Trujillo (2006), and Baleanu Diethelm, Scalas, and Trujillo (2012) for comprehensive reviews of the fractional calculus literature.

Drawing on the prior literature (Samko et al. 1993, Kilbas et al. 2006), Tarasov (2018) suggests that an economic process with power-law type memory can be captured by the following fractional integral equation:

$$I^{\alpha}X(t) := \int_0^t \frac{1}{\Gamma(\alpha)}(t-\tau)^{\alpha-1}X(\tau)d\tau\,. \tag{1}$$

In this equation, $X(t)$ represents the *input* of the economic process being modeled. The LHS is the *output*, where I^{α} is a non-integer fractional integral of order α $(\alpha > 0)$. The RHS of this equation is obtained based on the Reimann-Liouville fractional integral (Kilbas et al. 2006), which is a generalization of the standard n-th integral. $\frac{1}{\Gamma(\alpha)}(t-\tau)^{\alpha-1}$ is called the *kernel* of the fractional integral, which is interpreted as a *memory function* that is capable of capturing how the output at the present time depends on the history of the input up to the present time. $\Gamma(x)$, the Gamma function, for $x > 0$, has the following form:

$$\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}dt. \tag{2}$$

In the reminder of this section, we develop our repeat purchases model based on this fractional integral equation governing an economic process with memory.

3.2. Rate of Adoptions based on the Bass Model

To utilize fractional integral in Eq. (1), we first need to specify the input variable $X(t)$. For the product sales processes we are considering in this study, $X(t)$ represents the rate of adoptions (i.e., initial sales), and the output $I^{\alpha}X(t)$ captures the rate of sales, including both adoptions and repeat purchases.

The rate of adoptions can be defined by a diffusion model. Pioneering scholars have presented various diffusion models to estimate and predict diffusion of innovation in a population. Comprehensive reviews of these models have been provided by Meade and Islam

(1998) and Mahajan et al. (2000). Applying a bell-shaped growth curve to capture a noncumulative diffusion process is at the center of these endeavors. A mathematical realization of the bell-shaped diffusion process is represented by the seminal Bass Model (Bass 1969), which forms the foundation of the repeat purchases model we propose in this study.

The Bass Model stipulates that the noncumulative rate of adoptions of a product at time t, denoted by $y(t)$, and the cumulative number of adoptions, denoted by $Y(t)$, satisfy the following first-order differential equation:

$$\frac{dY(t)}{dt} = y(t) = pm + (q-p)Y(t) - \frac{q}{m}Y^2(t), t \geq 0, \tag{3}$$

where p and q are *coefficient of innovation* and *coefficient of imitation*, respectively, and m represents the size of the market potential. Note that Eq. (3) is an integer-order differential equation. Solving this equation with the initial condition $Y(0) = 0$ yields:

$$Y(t) = \frac{m(1-e^{-(p+q)t})}{(1+\frac{q}{p}e^{-(p+q)t})}, \tag{4}$$

$$y(t) = \frac{m(p+q)^2}{p}\frac{e^{-(p+q)t}}{(1+\frac{q}{p}e^{-(p+q)t})^2}. \tag{5}$$

The noncumulative version of the Bass diffusion curve, $y(t)$, effectively captures the bell-shaped growth pattern of adoptions, whereas the cumulative version $Y(t)$ exhibits the well-known S-shaped curve.

3.3. Integrating GDMR and Economic Process with Memory

Built on fractional calculus, or more specifically Eq. (1), and the Bass Model, we develop an extension of the Bass Model that is capable of capturing both adoptions and repeat purchases. In this new model, the sales rate at time t, $S(t)$, equals a fractional integral of the adoption rate, $y(t)$, which, in turn, equals the integral of the adoption rate times the memory function

($\frac{1}{\Gamma(\beta)}(t-\tau)^{(\beta-1)}$) that specifies how much past adopters contribute to repeat purchase at the present time. Formally, we have

$$S(t) = I^{\beta} y(t) = \int_0^t \frac{1}{\Gamma(\beta)} (t-\tau)^{(\beta-1)}\, y(\tau) d\tau, \quad t > 0, \tag{6}$$

where $y(t)$ is defined by the Bass Model, the new parameter β ($0 \leq \beta \leq 1$), which we term the *coefficient of repeat purchases*, is the only new parameter added to the Bass Model, and its value determines the proportion of repeat purchases. When $\beta = 0$, Eq. (6) reduces to the Bass Model. We call this extension model the *Generalized Diffusion Model with Repeat Purchases or GDMR*. It can be shown that, by adding repeat purchases to the adoptions, the GDMR generates asymmetrical bell-shaped sales growth curves that resemble the one illustrated in Figure 1.

To better understand the GDMR, let us take a further look at the memory function in the RHS of Eq. (6), i.e., $\left[\frac{1}{\Gamma(\beta)}(t-\tau)^{(\beta-1)}\right]$. Based on this memory-based interpretation, changes of the output variable (i.e., sales) in the GDMR at the current time depends on the value of the input variable (i.e., adoptions) both in the past and at the current time (i.e., $0 \leq \tau \leq t$). The RHS of Eq. (6) is a first-order integral of the rate of adoptions, i.e., $y(\tau)$, multiplied by the memory function, i.e., $\left[\frac{1}{\Gamma(\beta)}(t-\tau)^{(\beta-1)}\right]$. This means that, in calculating the rate of repeat purchases (and sales) at the current time t, the memory function assigns different weights to the past adoptions up to time t. In other words, although all past adoptions can lead to repeat purchases, they contribute to repeat purchases at different rates, as explained below.

Note that the memory (kernel) function in Eq. (6) is a power-law type function (Tarasov 2018), which assigns lower weights to earlier events and higher weights to more recent events, as illustrated in Figure 2. This implies that, on average, those who have adopted more recently are more likely to make repeat purchases than are those who have adopted earlier.[1] Although it may appear interesting at first glance, this phenomenon has ample empirical support and is theoretically intuitive. First, repeat purchases typically include upgrades within the same product line. There is empirical evidence suggesting that more recent buyers are more likely to upgrade to a newer product version than earlier buyers are. For instance, an analysis of sporting game sales data shows that buyers of more recent game versions are more likely to buy current year's version than are those who have bought older versions (Qu, Jiang, Lotfi 2018).

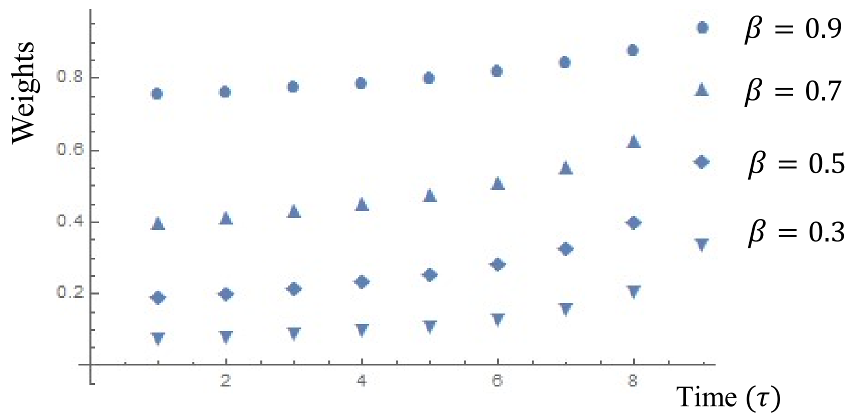


Figure 2. Memory Function Assigns More Weight to More Recent Adoptions (*t*=10)

Second, prior studies show that consumers' enthusiasm toward a product tends to die down as the product gradually turns old in their eyes (e.g., the declining interest in wearable activity trackers (Mobility 2014), smartphones, tablet computers, laptop computers, and TVs (Business Insider 2016)). The decline in adopters' enthusiasm and engagement results in a drop in their tendency to make multi-unit ownership and replacement purchases, thus leading to lower rate of repeat purchases for earlier adopters. Third, as explained before, repeat purchases refer to purchases of the original product or incrementally improved product versions/generations. When a new product-line built on innovation types other than incremental innovation (mainly radical

[1] In Appendix A we explain the memory characteristics of the GDMR.

radical) emerges, departures will likely occur. For example, the emergence of Blu-ray technology attracted consumers away from DVD players. Similarly, the growth in popularity of smartphones contributed to the decline in that of MP3 players (Statista 2019b). Given that the decline in interest in a focal product over time is largely due to developing interest in an alternative product, departures also suggest the pattern that, on average, earlier adopters contribute less to repeat purchases. Since the first, second, and third points mentioned above all suggest a lower (higher) rate of repeat purchases for earlier (more recent) adopters, the rate of repeat purchases is expected to follow a pattern governed by the power-law function.

This memory-based interpretation of the GMDR shows that, despite its parsimonious form, the model can capture the complex dynamics of repeat purchase processes, thus providing further theoretical justification for using fractional calculus to model repeat purchases.

3.4. Coefficients of Repeat Purchases and Repeat Purchase Scenarios

Recall that in the GDMR, the coefficient of repeat purchases (β) takes values between 0 and 1, i.e., $0 \leq \beta \leq 1$. In this subsection, we examine three repeat purchase scenarios, which can help us better understand the link between the frequencies of repeat purchases and the coefficient of repeat purchases (β).

We first consider a high repeat purchase scenario in which the expected *frequency of repeat purchase* (φ) by existing adopters equals 1, implying that, on average, each existing adopter makes one repeat purchase in each unit time. We call this scenario *periodic-repeat-purchases*. Under this scenario, the rate of repeat purchases at time t equals the cumulative number of adoptions just before time t. This number is added to the instantaneous rate of adoption exactly at time t to produce the sales rate at t. This means that the sales rate at time t equals the cumulative number of adoptions up to and including time t. Mathematically, this implies that the memory function equals 1 in the GDMR, which is achieved at $\beta = 1$. Therefore, under the periodic-repeat-purchases scenario, we have

$$S(t) = \int_0^t y(\tau)\, d\tau = Y(t) = I^{\beta=1} y(t). \tag{7}$$

In this case, $\beta = \varphi = 1$, and the noncumulative sales rate at time t equals the cumulative number of adoptions at time t, hence following an S-shaped curve.

We next examine another extreme scenario, in which no repeat purchase occurs ($\varphi = 0$), hence sales are composed of only adoptions. We call this the *adoption-only* scenario, under which the sales rate follows the noncumulative adoption rate defined by the Bass Model, which requires $\beta = 0$ in the GDMR:

$$S(t) = y(t) = I^{\beta=0} y(t). \tag{8}$$

In this case, the sales rate follows a symmetrical bell-shaped curve.

In most practical scenarios, the average frequency of repeat purchases likely falls between the two boundary scenarios described above, i.e., $0 < \varphi < 1$.[2] Specifically, existing adopters, on average, make *more than zero but less than one* repeat purchases per unit time. Integrating such a rate of repeat purchases into the instantaneous adoption rate yields a sales curve that falls between the S-shaped curve (with β =1) and the symmetrical bell-shaped curve (with β =0). For this *intermediate* scenario, we need to use the fractional integral of noncumulative adoption defined in Eq. (6), with $0 < \beta < 1$, to represent the sales rate. In this scenario, the coefficient of repeat purchases (β) increases with the frequency of repeat purchases (φ) and the exact rate at which adoptions at a given time in the past contribute to repeat purchases at the current time is determined by the memory (kernel) function in the RHS of Eq. (6).

[2] In rare cases where the average frequency of repeat purchases is greater than 1, there are two possible solutions: (i) redefine the unit of time so that the frequency falls below 1, or (ii) add another parameter to the GDMR so that the magnitude of a sales curve can be further adjusted. The details of the second approach is provided in Appendix C.

The dynamics of the GDMR under the three discussed scenarios are illustrated in Figure 3. In this figure, the bottom bell-shaped curve represents the adoption-only scenario ($\beta = 0$); the top S-shaped curve depicts the periodic-repeat-purchases scenario ($\beta = 1$). In between the two boundary curves are intermediate curves corresponding to $0 < \beta < 1$. Increasing the value of β causes the sales curve to shift away from the symmetrical bell-shaped curve corresponding to $\beta = 0$ and move closer to the S-shaped curve corresponding to $\beta = 1$, implying a higher sales rate due to more repeat purchases.

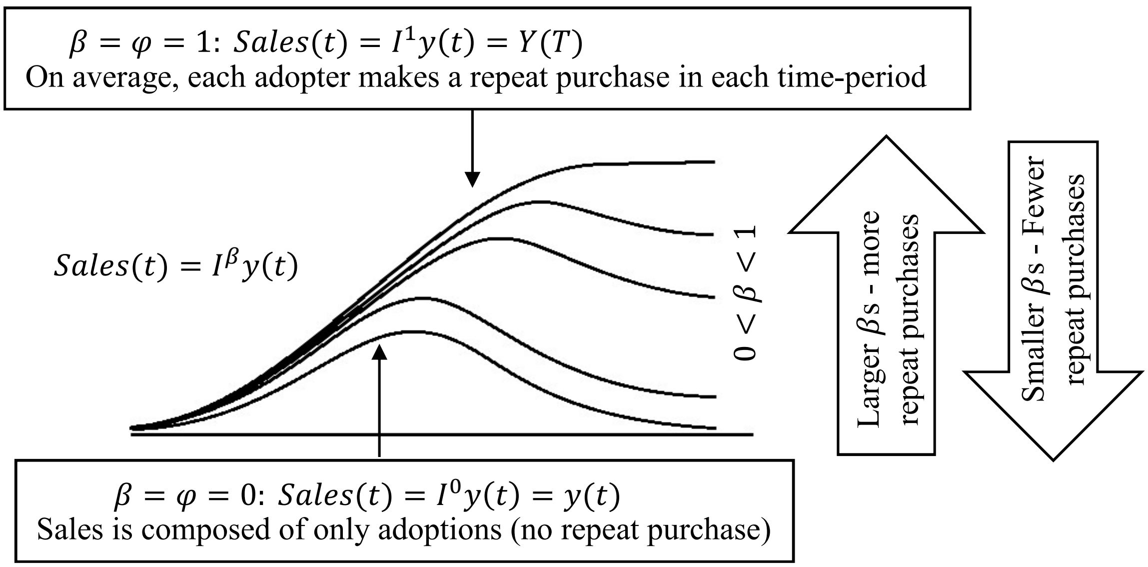


Figure 3. Three Scenarios under the GDMR

Built on a fractional integral, Eq. (6) defines the sales rate with repeat purchases. The rate of repeat purchases at time t, denoted by $R(t)$, is the difference between sales rate and adoption rate at t:

$$R(t) = S(t) - y(t) = I^\beta y(t) - y(t). \tag{9}$$

As shown in Eqs. (3) and (5), the adoption rate, as defined in the Bass Model, depends on the adoption parameters p, q, and m, and does not change with β. The sales rate $S(t)$, as shown in

Figure 3, increases with β. Therefore, the rate of repeat purchases defined in Eq. (9) is an increasing function of β.

3.5. Approximation and Dynamics of the GDMR

Estimating parameters based on Eq. (6) may present difficulties due to the complexity of computation associated with the fractional integral operator. Therefore, we need a mathematical operator that has the desirable properties of Eq. (6) and is computationally feasible. We introduce the operator $I_{n,k}^{\beta}$ instead of I^{β} and reformulate the GDMR as:

$$S(t) = I_{n,k}^{\beta} y(t), \tag{10}$$

where n and k are parameters of the approximate operator. Increasing the values of n and k results in $I_{n,k}^{\beta}$ converging to I^{β}. The detailed derivations of the approximate operator and its convergence to the original one are demonstrated in Appendix B.

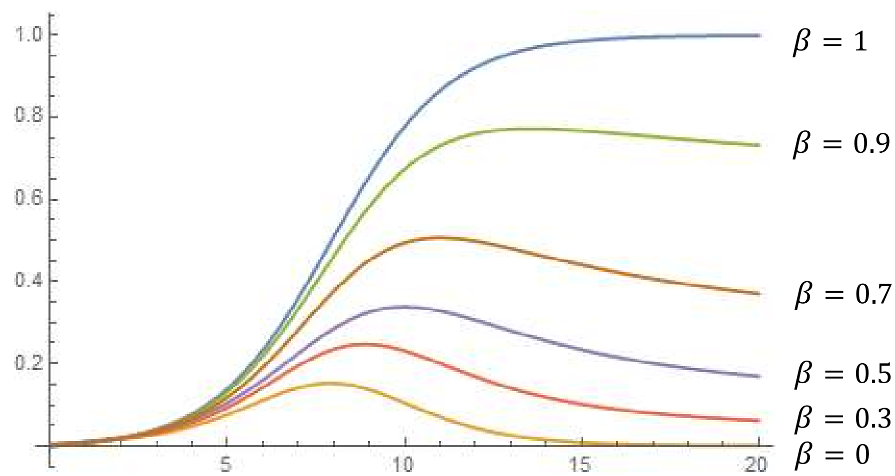


Figure 4. Dynamics of GDMR with Respect to β (p = 0.005, q = 0.6, and m = 1)

Based on the operationalization presented above, we conduct numerical analysis to examine how the rate of repeat purchases change with the coefficient of repeat purchases (β). The results are summarized in Figure 4, which illustrates how the GDMR sales curve varies with β (values of other parameters are fixed at p = 0.005, q = 0.6, and m = 1).

Figure 4 provides a clearer picture how the sales curve changes with the value of the coefficient of repeat purchases (β). Consistent with our previous theoretical predictions, a higher β value in the GDMR is linked to a higher sales curve, thus implying a higher level of repeat purchases. As the value of β increases from 0 to 1, the sales curve moves further away from the bell-shaped curve with no repeat purchases and closer to the S-shaped curve representing one repeat purchase per unit time by existing adopters.

3.6. Time to Peak Sales

Based on the provided operationalization, we can also derive the time of peak sales, t^*, for the GDMR by finding the root of the first derivative of sales as:

$$\frac{d}{dt}(I^{\beta}y(t)) = 0, \tag{11}$$

where $\frac{d}{dt}I^{\beta}y(t)$ is the Riemann–Liouville fractional derivative of order $1-\beta$ of $y(t)$ (Kilbas et al. 2006). In our model extension, t^* is the root of the Riemann–Liouville fractional derivative of order $1-\beta$ of the noncumulative adoption trend, whereas in the Bass Model, it is the root of the first order derivative of noncumulative adoption curve that leads to the peak point of adoption.

3.7. Incorporating Marketing Mix Variables

It is well understood that marketing mix variables (e.g., price and advertising) can affect a diffusion process. Likewise, it is expected that marketing mix variables can influence the magnitude of repeat purchases and subsequently that of the total sales. In this section, we examine how marketing mix variables can be incorporated into the GDMR.

Prior research has explored various ways of accounting for the influence of marketing mix variables on the diffusion of products (Bass, Jain, and Krishnan. 2000). Here, we implement the approach used in the Generalized Bass Model (GBM) (Bass et al. 1994). Specifically, we incorporate the marketing mix variables into cumulative sales by the GDMR as:

$$CumS(t) = I^{\beta+1}[y(X(t))], \tag{12}$$

where *CumS*(t) is cumulative sales, y is periodic adoptions, and $X(t)$ is the *cumulative marketing effort* as defined in the GBM by Bass et al. (1994). For instance, including price (*pr*) and advertising (*Adv*), $X(t)$ takes the following form:

$$X(t) = t + \gamma * ln\left[\frac{Pr(t)}{Pr(0)}\right] + \delta * ln\left[\frac{ADV(t)}{ADV(0)}\right], \tag{13}$$

where γ and δ are coefficients capturing the effects of changes in price and advertising, respectively. To differentiate it from the GDMR without marketing mix variables, we name the version with marketing mix variables the *Generalized Diffusion Model with Marketing Mix Variables* (GDMRX). GDMRX in its noncumulative form can be shown as:

$$S(t) = x(t) * I^{\beta}[y(X(t))], \tag{14}$$

where $x(t)$ is *noncumulative marketing effort* as defined in the GBM.

4. Empirical Analysis

In this section, we empirically evaluate how well the GDMR performs compared to alternative models, how incorporating marketing mix variables influences model performance, and how the availability of separate datasets for adoptions and repeat purchases can further guide parameter estimation.[3]

4.1. Benchmark Models

Because the GDMR captures all components of sales including adoptions, replacement purchases, and multi-unit purchases, and all three sales components are expected to be present in

[3] Sample source code for the empirical analysis is available from the authors upon request.

our datasets, we need to compare the GDMR with alternative models that also include adoptions, replacement purchases, and multi-unit purchases. However, unlike the broader literature on product adoption, the literature concerning repeat purchases is rather limited, and we could not find readily available models that explicitly incorporate the three components of sales. Therefore, we choose to develop our own benchmarks based on models proposed in the prior literature.

For comparison purposes, we consider two benchmark sales models that incorporate initial purchases, replacements, and multi-unit ownerships. The first model incorporates the Bass Model for initial purchases, the Kamakura and Balasubramanian (KB) (1987) replacements model, and the Bayus et al. (BHL) (1989) multi-unit purchases model. For expositional convenience, we name this model the *Bass-KB-BHL Sales Model*. The second benchmark model is composed of the same initial purchases model and replacement model, but uses a different multi-unit purchase model developed by Steffens (2003). We name this model the *Bass-KB-Steffens Sales Model*. To help readers better understand the two benchmark models, we next briefly elaborate on the KB Model, BHL Model, and Steffens Model.

Kamakura and Balasubramanian (1987) formulate a product replacement model for durable products based on two assumptions: (i) a product is immediately replaced after it fails to perform up to users' expectations, and (ii) these failures can be represented by a probability distribution function over all product units. The model can be expressed as:

$$r(t) = \sum_{i=1}^{t-1}[y(i) + r(i)]\,[Sur(t-i-1) - Sur(t-i)], \tag{15}$$

where $[y(i) + r(i)]$ represents sales at time i and is composed of initial purchases, $y(i)$, and product replacements, $r(i)$.[4] $Sur(\tau)$ is a survival function capturing the probability that a product

[4] The model assumes that ownership of multiple units is insignificant.

unit fails after τ. For example, with truncated normal distribution as the distribution of survivals,[5] $Sur(\tau)$ takes the following form:

$$Sur(\tau) = \frac{\Phi(\frac{w\tau}{L} - h)}{\Phi(-h)}, \tag{16}$$

where $w = h + \phi(-h) / \Phi(-h)$, $\phi(.)$ is the standard normal probability density function, and

$$\Phi(x) = \int_{z=x}^{\infty} \phi(z)dz. \tag{17}$$

Bayus et al. (1989) develop a model that explicitly considers multi-unit ownerships based on the premise that the older is the product unit in use, the more likely is the purchase of an additional unit. The proposed functional form is

$$mul(t) = \sum_{i=1}^{t-1} siu_{i,t-1}\, g(t-i), \tag{18}$$

where $mul(t)$ represents the number of multi-unit purchases at time t, $g(\tau)$ is the hazard rate for multi-unit purchases at τ, and

$$siu_{i,t} = [y(i) + r(i) + mul(i)]Sur(t-i). \tag{19}$$

In this model, $g(\tau)$ is empirically determined by Bayus et al. (1989). Following Steffens (2003), we adopt a logistic growth function for $g(\tau)$:

$$g(\tau) = \frac{\delta(1-e^{-\alpha})}{(1+\beta e^{-\alpha})}. \tag{20}$$

Steffens (2003) views the purchase of an additional unit as a diffusion process and formulates it based on the Bass Model:

$$\frac{\mathrm{d}mul(t)}{\mathrm{d}t} = (\pi Y(t) - mult(t))(a + b * mult(t)), \tag{21}$$

where $mult(t)$ is the cumulative number of first additional units purchased by users, $Y(t)$ is the cumulative adoptions representing the upper limit for multi-unit ownerships, a represents the external influences on first multi-unit adoptions, and b represents the word-of-mouth influences

[5] There is extensive support in the literature for using truncated normal distribution for this purpose (Steffens 2003).

on first multi-unit adoptions. Steffens suggests that the adoption of more additional units by users (e.g., second additional units, third additional units) can be similarly modeled. The Steffens Model does not have a closed-form solution and can only be numerically integrated.

By summing up the aforementioned models for initial, replacement, and multi-unit ownership purchases, we obtain two benchmark models: Bass-KB-BHL and Bass-KB-Steffens.

4.2. Comparison with Benchmark Models Using Only Sales Data

We use three aggregate sales datasets to evaluate the model fitting and forecasting performance of the GDMR and the two benchmarks. The datasets include annual sales of Notebook Computers from year 2005 to 2014 (Morgan Stanley 2015), PC total global annual sales from year 2006 to 2015 (Statista 2014), and iPad sales from the third quarter of year 2010 to the second quarter of 2017.[6] All three datasets include only aggregate sales data, and a breakdown into adoption purchases, replacement purchases, and additional-unit purchases is unavailable.

Following the parameter estimation approach by Srinivasan and Mason (1986), we use the changes in the cumulative number of adoptions between two consecutive periods to represent the noncumulative adoptions trend in the benchmark models.

For the first two datasets, due to potential left-hand truncation, we add an intercept, s_0, to both the GDMR and the benchmark models to help improve model estimation. In our application of the KB replacement model, we assume that sales include additional-unit purchases as well (i.e. sales in Eq. (15) is of the form $[y(i) + r(i) + mul(i)]$ instead of $[y(i) + r(i)]$). Because our test data is for consumer electronics, to guide the estimation of the benchmarks' replacement component, we assume that the average life of the products is less than or equal to eight years.

[6] Obtained from Apple's quarterly summaries. For instance, 2013 Q4 data is found in https://www.apple.com/newsroom/pdfs/q4fy13datasum.pdf. Starting from the third quarter of 2010, we sum four quarters of sales to derive one year worth of iPad sales.

Table 1. GDMR Parameter Estimation for Notebook Computer Sales

	Estimate	Standard Error	t-statistics	P-value
β	0.46	0.110	4.9	0.00
p	0.019	0.005	4.5	0.01
q	0.627	0.108	5.7	0.00
m	4.6×10^{8}	1.5×10^{8}	3.2	0.02
s_0	4.8×10^{7}	9.9×10^{6}	4.9	0.00

Table 2. GDMR Parameter Estimation for PC Sales

	Estimate	Standard Error	t-Statistics	P-value
β	0.24	0.130	5.70	0.00
p	0.014	0.009	1.56	0.18
q	0.725	0.201	3.60	0.02
m	4.8×10^{8}	2.0×10^{8}	2.35	0.07
s_0	2.3×10^{8}	1.7×10^{7}	13.44	0.00

Table 3. GDMR Parameter Estimation for iPad Sales

	Estimate	Standard Error	t-Statistics	P-value
β	0.48	0.018	28.44	0.00
p	0.05	0.003	18.18	0.00
q	1.17	0.046	25.36	0.00
m	1.53×10^{8}	5.5×10^{6}	27.70	0.00

The GDMR parameter estimates for the three products are summarized in Tables 1, 2, and 3, respectively. As we can see from the tables, except for parameter p for PC sales, all other parameter estimates for all three products are statistically significant, indicating a good overall model fit in all three cases. The value of the coefficient of repeat purchases (β) ranges from 0.24 to 0.48 indicating that sales curves for these three products lie between the bell-shaped and S-shaped curves. Therefore, the traditional diffusion parameter values for p, q, and m would have been biased without explicitly incorporating repeat purchases in the model formulation.

Furthermore, the estimate for s_0 is statistically significant for the first two products, showing that it is an effective way to addressing the left-hand data truncation issue.

We also compare the GDMR against the two benchmark models using the three datasets; their model fitting and forecasting performance are summarized in Table 4. The fitting accuracy is measured in terms of R^2 and MAPE (mean absolute percentage error). From the summary, it is clear that the GDMR, despite having fewer parameters, consistently leads to better fits than do the two benchmark models.

Table 4. Comparison of Fit and Forecast for GDMR, Bass-KB-BHL, and Bass-KB-Steffens

		Full data fit		Forecast	
				Years Ahead MAPE	
		R^2	MAPE	One	Two
Notebook Computer	GDMR	0.9995	1.68	14.46	8.79
	Bass-KB-BHL	0.9054	10.09	6.83	31.09
	Bass-KB-Steffens	0.9573	5.40	4.64	25.75
PC	GDMR	0.9995	2.08	0.94	8.84
	Bass-KB-BHL	0.6731	5.25	10.28	8.35
	Bass-KB-Steffens	0.8390	3.11	10.28	18.11
iPad	GDMR	0.9999	1.34	3.10	1.90
	Bass-KB-BHL	0.7763	15.31	14.55	37.84
	Bass-KB-Steffens	0.8439	9.05	14.48	37.62

The forecasting performance of the three models is evaluated based on the MAPE of one-year- and two-years-ahead forecasting. The results in Table 4 show that, with the exception of one-year-ahead forecast for notebook computers and two-years-ahead forecast for PCs (for which BassKB_Steffens and Bass-KB-BHL respectively are better than the GDMR), the forecasting performance of the GDMR is also better than that of the benchmark models.

The model fitting results for the GDMR, Bass-KB-BHL, and Bass-KB-Steffens are depicted in Figures 5, 6, and 7 for the three datasets. To contrast these repeat purchase models with the adoption-only Bass Model, we include the fitted Bass diffusion curve in these figures as well. For iPad sales, the GDMR appears to fit the data better than the benchmarks after the sales process peaks. This can be attributed to the fact that while the GDMR does account for the decline-over-time in the rate of repeat purchases, while the benchmarks do not. As expected, the Bass Model fits the data reasonably well before the peak point of sales, but declines much more quickly than the other three models after the peak, resulting in poor model fits. This is particularly obvious in Figure 7 for iPad sales, where the fitted Bass Model curve is clearly off. Therefore, the Bass Model is not recommended when repeat purchases cause the aggregate sales to exhibit an asymmetric curve. For this reason, the Bass Model is excluded from the rest of the performance comparisons.

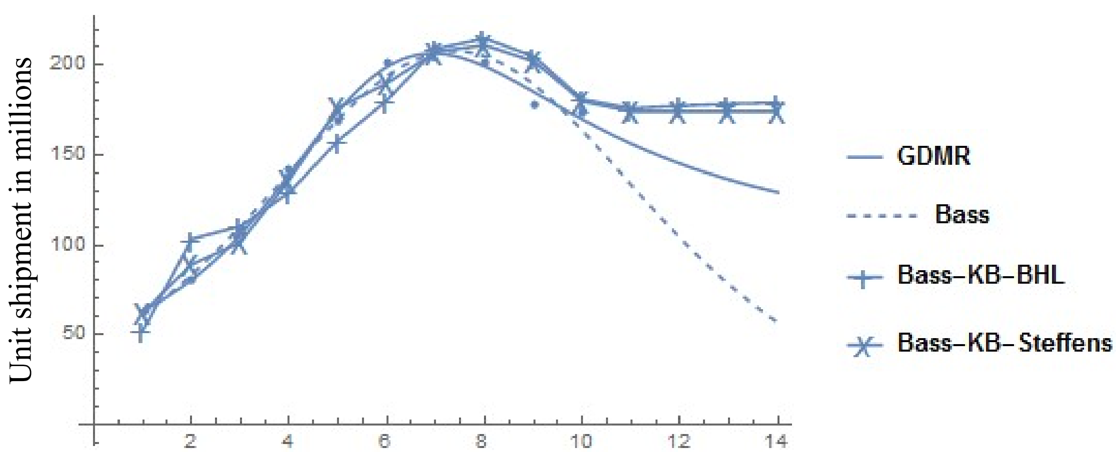


Figure 5. Comparison of Model Fitting for Notebook Computer Sales Data 2005–2014

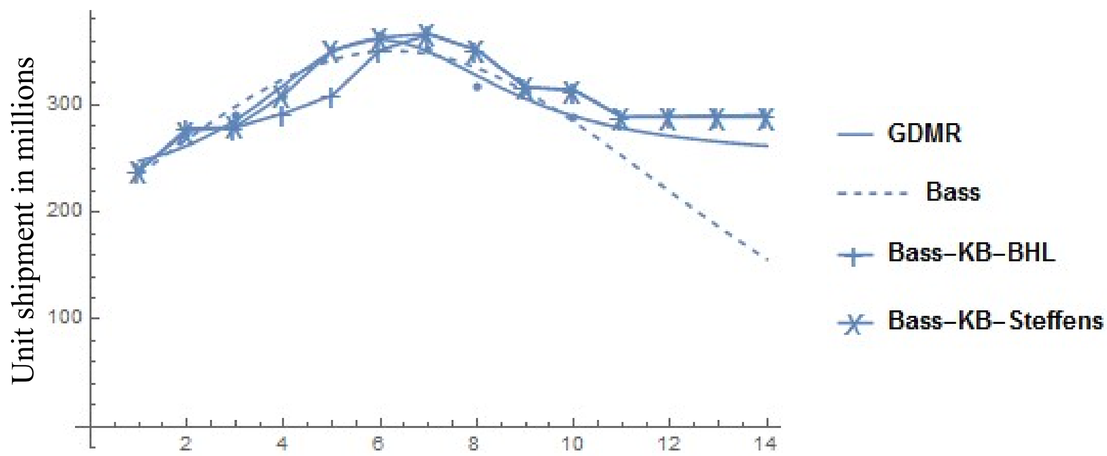


Figure 6. Comparison of Model Fitting for the Global PC Sales Data 2006–2015

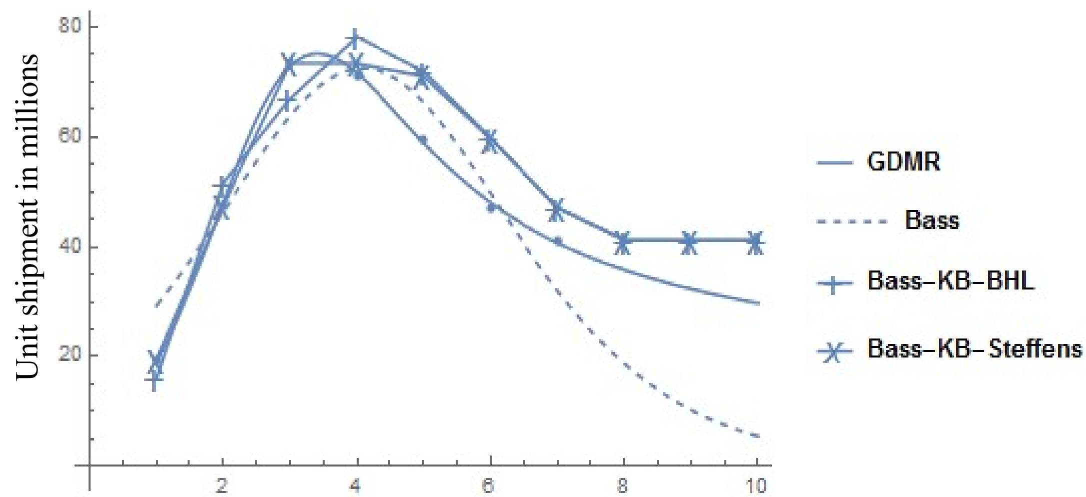


Figure 7. Comparison of Model Fitting for iPad Sales Data (Q3 of 2010 to Q2 of 2017)

With all results considered, we conclude that the GDMR is a better model for both model fitting and forecasting.

4.3. Empirical Analysis with Marketing Mix Variables

We use iPod sales from 2004 to 2014, obtained from Apple's quarterly summaries, to illustrate the performance of the proposed model with marketing mix variables included (see Eq. (13)). To obtain the average selling price of the iPod, we divide the revenue generated by iPod sales by the corresponding number of units sold. The comparison of model fitting between the GDMR and the GDMRX is shown in Figure 8. The parameter estimates and model fitting measures are summarized in Table 5.

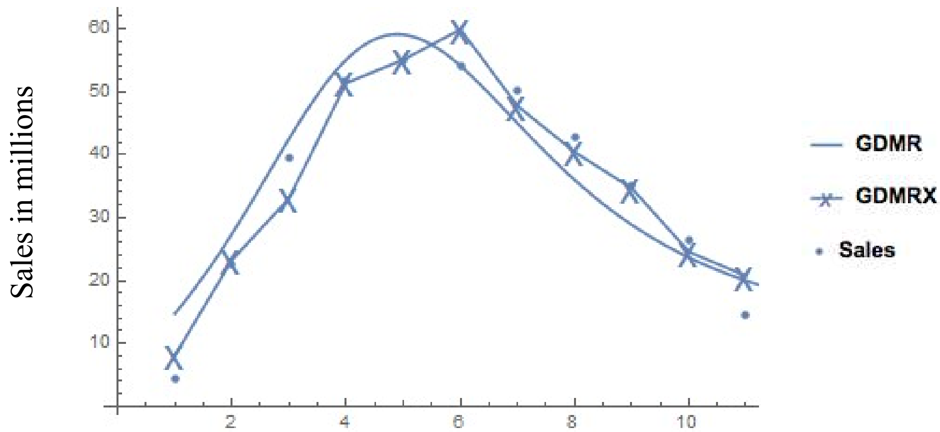


Figure 8. GDMR and GDMRX for the iPod Sales Data

Table 5. GDMR and GDMRX Parameter Estimates, R^2, and SSE for iPod Sales

	β	p	q	m	γ	R^2	SSE
GDMR	0.278	0.037	0.656	2.18×10^8	-	0.9998	9.8×10^{13}
	(0.066)	(0.004)	(0.066)	(0.33×10^8)			
GDMRX	0.21	0.027	0.603	2.56×10^8	–0.745	0.9998	8.4×10^{13}
	(0.096)	(0.01)	(0.065)	(0.58×10^8)	(0.787)		

Note: SSE, sum of squared errors; values in parentheses represent standard errors. Estimation results are based on fitting the cumulative forms of the GDMR and GDMRX to cumulative sales data.

These results show that both the GDMR and the GDMRX perform well on the iPod data with the GDMRX having a slight edge in terms of sum of squared errors (SSE). This is not surprising given that the GDMRX is a more flexible model than the GDMR.

4.4. When Both Sales Data and Adoptions Data are Available

Ideally, the benchmark models should be used in cases for which separate datasets for each component of sales is available. The Notebook Computer, PC, and iPad datasets we have used record only aggregate sales. As a result, we can fit only the summations of the three components to the sales data. This treatment allows for greater flexibility, but could increase the risk of overfitting and result in the predicted trends diverging from the theoretically correct trends. Judging by the predicted and actual sales trend, the GDMR performs well on aggregate sales data; however, we are not able to assess whether the adoption trend predicted by the GDMR closely matches the true adoption trend.

What if both adoption data and sales data are available? To answer this question, we compare the GDMR with the benchmark models using separate adoption data and sales data for DVD players in the U.S. from year 1997 to 2018. The adoption data from 1997 to 2007 is derived by multiplying the penetration of DVD in the U.S. households reported by the Consumer Technology Association (Uncommon Wisdom Daily 2015) by the population of households in the U.S. (Statista 2016). The corresponding sales data from 1997 to 2010 is reported by the Digital Entertainment Group (DEG) and from 2014 to 2018 is reported by Statista.[7]

Our test using this data includes three steps. First, we fit the Bass Model to the noncumulative adoption data to estimate adoption parameters p, q, and m. Second, we treat the resulting adoption parameters as known and then fit the GDMR to the sales data to estimate the coefficient of repeat purchases β. We withhold five last sales data points (2014-2018) for evaluating the GDMR's performance in forecasting sales. The parameter estimates for the Bass Model and the GDMR (only β is estimated by the GDMR with p, q, and m being estimated by the Bass Model and entered into the GDMR), as summarized in Table 6, are mostly significant. Third, a similar estimation procedure is repeated for the benchmark models.

[7]There is some inconsistency in the two sets of data in that adoption is reported to be slightly larger than sales from 1998 to 2000 and that adoption is zero in 1997, whereas the corresponding sales is non-zero. The inconsistency, however, has a negligible impact on parameter estimation. Therefore, we choose to keep the entire sample period for parameter estimation.

Table 6. Parameter Estimates Based on DVD Player Adoptions and Sales (Two-Step Procedure)

Model	R^2	SSE	Parameter	Estimate	Standard error	t-statistics	P-value
Bass	0.9246	1.0×10^{14}	p	0.003	0.002	1.3	0.23
			q	0.769	0.130	5.9	0.00
			m	9.9×10^{7}	1.3×10^{7}	7.7	0.00
GDMR	0.9923	6.6×10^{13}	β	0.48	0.01	46.4	0.00

Note: SSE, sum of squared errors.

The comparison of fits and forecasts obtained from the GDMR and the benchmark models is reported in Table 7 and depicted in Figure 9. From the table, it is clear that the GDMR outperforms the two benchmarks in terms of model fitting and forecasting. From the figure, we can see that, once again, the GDMR curve matches the actual sales trend better than do the benchmark models. The GDMR's forecasting performance for 2014-2018 sales is also much better than those of benchmark models. Bass-KB-Steffens shows an unrealistic increase in sales after the decline. This can be attributed to the joint estimation of the replacement and multi-unit ownership models using only the total repeat purchases data, which may result in biased parameter estimates for the benchmark models. Steffens (2003) also reports problems when replacements and multi-unit ownerships are jointly estimated.

Table 7. Comparison of GDMR and the Benchmarks Based on DVD Player Sales Data

	GDMR	Bass-KB-BHL	Bass-KB-Steffens
Fit R^2	0.9923	0.8057	0.9652
Fit MSE	4.7×10^{12}	3.11×10^{13}	5.6×10^{12}
Forecast MAPE	8.74	48.87	74.52

Note: MSE – mean square error; MAPE – mean absolute percentage error.

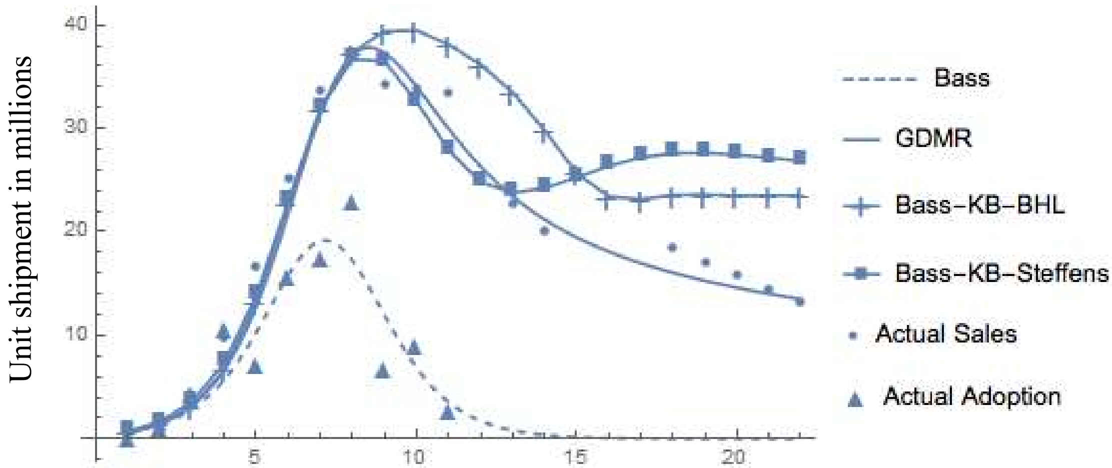


Figure 9. GDMR and Benchmarks along with the Bass Diffusion Curve Based on DVD Players Adoption and Sales (1997–2018)

Note that the parameter values summarized in Table 6 and model fitting shown in Figure 9 are obtained by first fitting the Bass Model to the adoption data and subsequently the GDMR to the sales data. This *two-step procedure* is expected to produce more reliable adoption parameter (m, p, and q) values, but is unfortunately feasible only if the adoption data is available. In another test, we assume that, as in the previous cases, the adoption data is unavailable, and we fit the GDMR directly to the sales data from 1997 to 2010 to obtain both adoption and repeat-purchase parameter values. For convenience, we term this the *GDMR-sales procedure*. We summarize the result of sales procedure based on fitting the GDMR to sales data from 1997 to 2010 in Table 8. By comparing the adoption parameter estimates in Tables 6 and 8, it is evident that the values of these parameters are remarkably similar, showing that the GDMR can effectively recover the adoption trend from sales data.

Table 8. Parameter Estimates Based on DVD Player Sales (GDMR-Sales Procedure)

β	p	q	M	R^2	SSE
0.47	0.005	0.7	1.0×10^8	0.9967	2.8×10^{13}
(0.06)	(0.001)	(0.062)	(1.4×10^7)		

Note: SSE, sum of squared errors; values in parentheses represent standard errors.

The two-step procedure and the GDMR-sales procedure's model fitting performances are shown in Figure 10. From the figure, the adoption trend and the sales trend produced by the two procedures are highly consistent, further demonstrating the power of the GDMR.

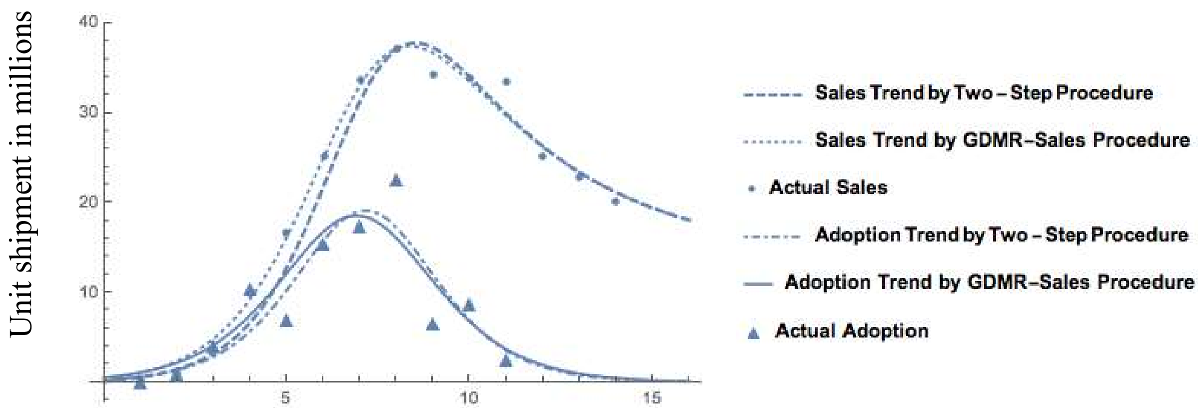


Figure 10. Comparison of Model Fitting for DVD Players Adoptions and Sales (1997–2018)

Furthermore, Figure 10 presents clear empirical evidence that repeat purchases can account for a growing proportion of total sales as a product continues to penetrate a market. Unless the repeat purchases are explicitly modeled, parameter estimates are likely to be biased, and forecasting performance could suffer. Therefore, a model such as the GDMR that incorporates repeat purchases should be adopted whenever repeat purchases are significant.

Based on the DVD players dataset, we also estimate the two benchmark models using a procedure similar to the GDMR-sales procedure to examine how accurately the benchmark models can recover the underlying adoption trend. We first fit the benchmark models directly to the sales data to obtain all adoption and repeat-purchase parameter values. We then compare the adoption trend estimated by the benchmark models with the adoption trend estimated by directly fitting the Bass Model to the adoption data. Results for the estimated sale and the corresponding adoption estimated by the Bass-KB-BHL Model and the Bass-KB-Steffens Model are shown in Figures 11 and 12, respectively. We can observe that the adoption trends estimated by the two benchmark models are considerably different from the adoption trend estimated by directly fitting the Bass Model to the adoption data. By comparing Figures 11 and 12 against Figure 10, we can see that, despite having fewer parameters, the GDMR performs better than the benchmark models in recovering the adoption trend based on the DVD player sales data. This offers a strong empirical evidence that the GDMR can more reliably separate adoptions from repeat purchases than the benchmark models do.

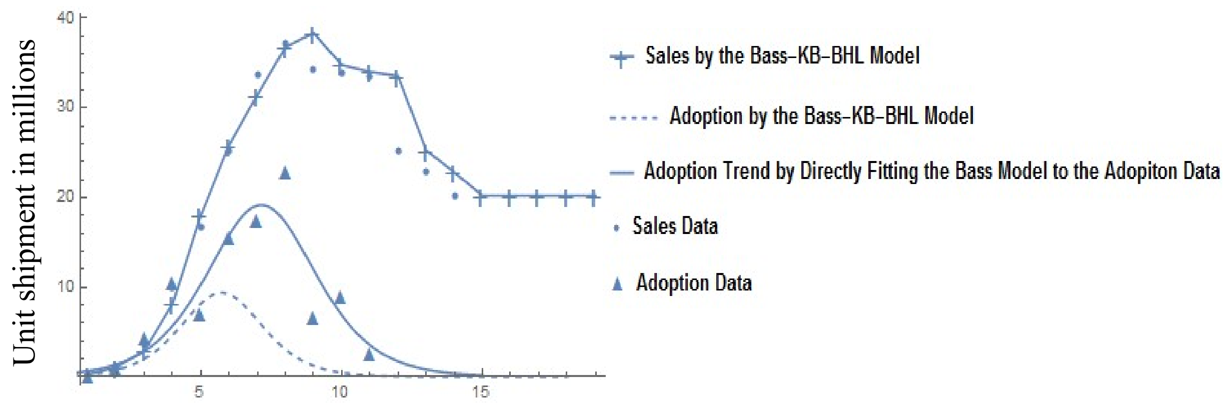


Figure 11. Sales Components Estimated by the Bass-KB-BHL Model

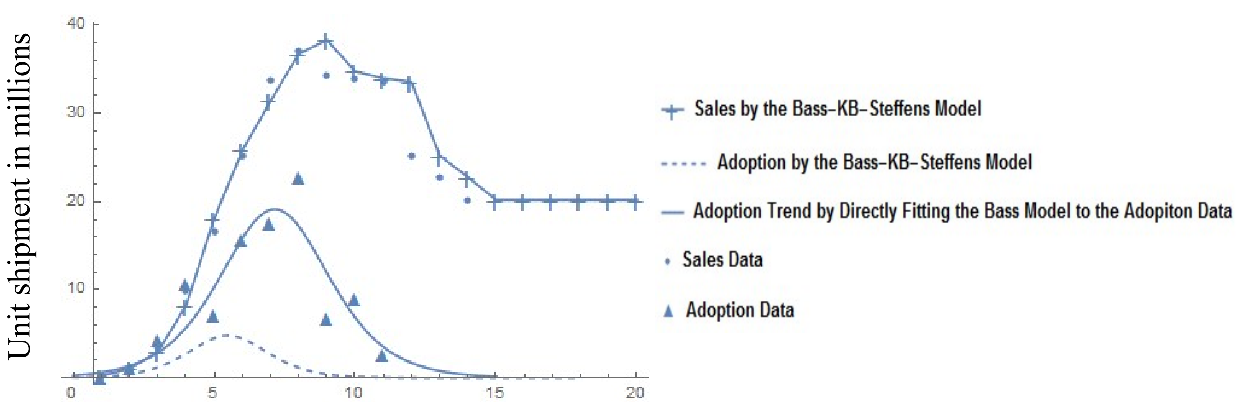
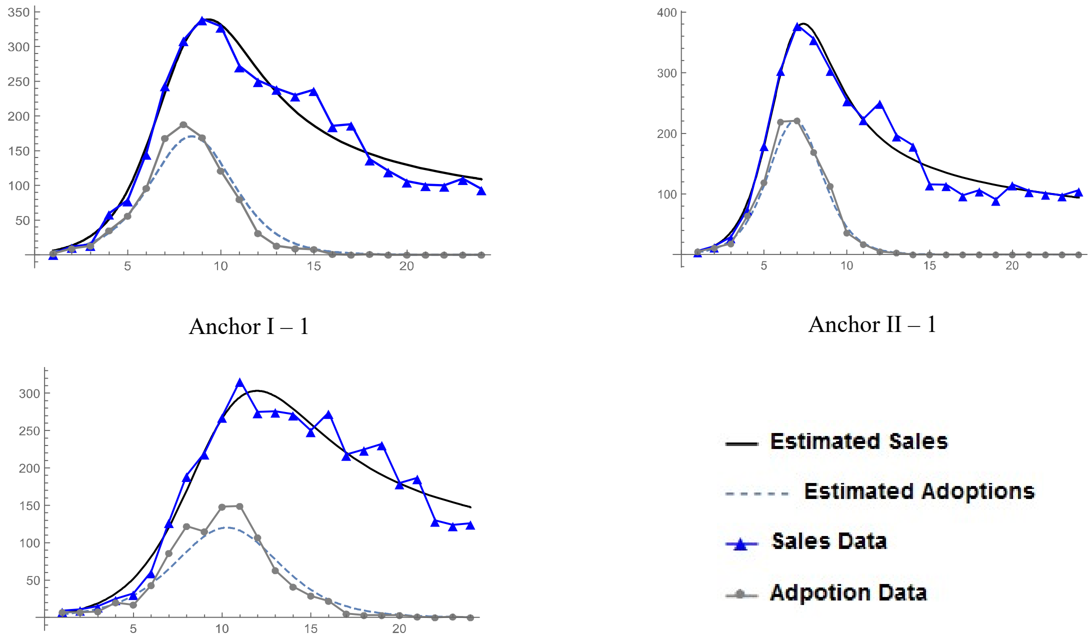


Figure 12. Sales Components Estimated by the Bass-KB-Steffens Model



Anchor I – 1

Anchor II – 1

Anchor III – 1

Figure 13. GDMR's Fit on Sales and Adoptions Based on Simulated Data

Ideally, we should evaluate the GDMR's capability in recovering adoption trend from sales data using more than one real dataset. In reality, however, it is extremely rare to have both adoption and sales data available for the same product. Therefore, we conduct extensive simulations to generate adoption and sales data for a wide variety of products. We subsequently repeat the aforementioned GDMR-sales procedure using the simulated data. Details about the simulation and the additional empirical analysis are provided in Appendix D. For the purpose of illustration, we select three representative cases and show them in Figure 13. The results again show that the GDMR is able to reliably separate repeat purchases from adoptions even when only aggregate sales data is used, thus further establishing the validity of the GDMR.

5. Conclusions and Discussions

This study aims to develop a generalized diffusion model to capture sales of a product with repeat purchases. By treating the sales process as an *economic process with memory*, we are able to utilize a branch of mathematics called fractional calculus, and develop a novel diffusion/sales model, the *Generalized Diffusion Model with Repeat Purchases* (GDMR), to account for repeat purchases that have become increasingly prevalent in today's markets. The GDMR generalizes the classic Bass Model using an integral operator with fractional order, and produces sales growth curves that stays above the adoption trend.

By assigning different values to the newly introduced parameter named *coefficient of repeat purchases*, which determines the value of the memory function that controls the rate of repeat purchases, the proposed GDMR can cover a wide continuum of sales growth scenarios ranging from symmetrical bell-shaped sales curves to S-shaped sales curves. This represents sales for a wide variety of products ranging from perfectly durable goods that are not subject to obsolescence and offer no incentive for repeat purchases, to products that have a short lifetime, multiple versions, or a high chance of becoming obsolete as a result of frequent releases of product upgrades.

Theoretically, the GDMR has several important advantages over alternative models. By clearly differentiating adoptions from subsequent repeat purchases, the GDMR overcomes the parameter estimation bias resulting from force-fitting the adoptions-only Bass Model to sales data that is typically "contaminated" by repeat purchases. Reducing the bias in parameter estimation can lead to better understanding of the product diffusion process and more accurate prediction of future sales.

As opposed to the models in the extant literature, the GDMR accounts for the decline in the rate of repeat purchases due to users' falling interest in the product over time and departures to new products. In addition, the GDMR does not exclude either replacements or multi-unit ownership purchases; therefore, it has broader applications compared to models that incorporate only one of these two sales components. The GDMR remains a parsimonious model, as it incorporates repeat purchases by adding only one parameter to the Bass Model. Furthermore, the GDMR can incorporate marketing mix variables such as pricing and advertising, thus making it more useful in helping businesses make better marketing and promotion decisions.

Empirically, we show that the GDMR is easy to implement and delivers superior performance when (a) only aggregate sales data is available, (b) data for marketing mix variables is available, and (c) data for adoption and sales are both available. In particular, the demonstrated consistent performance based solely on aggregate sales data is a quality unseen from other models proposed in the literature, and one that is of utmost importance as separate data for distinct sales components is rarely available. In addition, we find that the GDMR can effectively recover adoption trends when only sales data is available. Such robust performances validate that GDMR is a great choice in helping understand and predict product adoptions and sales trends for a long time frame.

We would also like to point out that the GDMR's contribution is not limited to the diffusion of innovations literature. By accounting for repeat purchases using a fractional integral of the adoptions trend, the GDMR presents a new interpretation of fractional integral, thereby contributing to the broad literature of fractional calculus in applied mathematics and science. It is very well known that integer-order integrals and derivatives have simple and clear geometric and physical interpretations, thus guiding numerous applications of these tools in science (Podlubny 2002). Interpreting fractional integrals and derivatives, however, has proven challenging. For more than 300 years since the introduction of fractional calculus, no clear geometric and physical interpretation of fractional derivatives and integrals has been introduced (Podlubny 2002). In the first international conference on fractional calculus held in New Haven (USA) in the year 1974, discovering the physical and geometric interpretations of fractional calculus was included in the list of open problems (Ross 1975). Only in more recent years, different interpretations of fractional calculus have emerged, including the economic interpretation proposed by Tarasova and Tarasov (2017). By proposing an interpretation of fractional integral in the context of product adoptions and repeat purchases, the present research represents the newest contribution to the cross-disciplinary endeavor of interpreting fractional calculus.

The present research, the fractional calculus-based model extension in particular, lays a foundation for future work in the diffusion of innovation domain. For instance, multigeneration diffusion models (e.g., Norton and Bass 1987, Jiang and Jain 2012) could be used as the base for the type of extension presented in this paper. In addition, the modeling approach implemented in this study, utilizing a spectrum of functions that run between a probability density function and its corresponding cumulative distribution function to capture the variations of a phenomenon of interest, is worth further examination. This approach may find more applications in other branches of business and economics research.

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Appendix A: Characteristics of the GDMR

In this Appendix, we mathematically demonstrate the following characteristics of the GDMR.

(i) The Reimann-Liouville fractional integral used in the formulation for the GDMR defines sales as a process with memory of adoptions. Adoptions from the past that are *remembered* at the current time result in repeat purchases at the current time.

(ii) The memory function used in the formulation for the Reimann-Liouville fractional integral (used in the GDMR) is a fading memory or power-law memory, meaning that it remembers the more recent adoptions better than the older ones. This means that based on the fading or power-law memory, those who have made their adoptions more recently are more prone to making repeat purchases at the current time.

(iii) lower βs correspond to lower memories of the process, meaning that under lower βs, adopters are less prone to making repeat purchases.

To demonstrate the abovementioned three characteristics of the GDMR, we view sales as an economic process with a memory of adoptions. In its general form, an economic process with memory can be shown as:

$$Y(t) = \int_0^t M(t,\tau) X(\tau) d\tau, \tag{A1}$$

in which $Y(t)$ is an endogenous variable (sales in the GDMR) associated with the exogenous variable $X(\tau)$ (adoptions in the GDMR) based on a linear Volterra operator (Tarasova and Tarasov 2018). The Volterra operator in Eq. (A1) is:

$$(.) = \int_0^t M(t,\tau)(.) d\tau, \tag{A2}$$

where $M(t,\tau)$ is a function called the memory function (Tarasova and Tarasov 2018). To describe the memory property of the operator shown in Eq. (A2) consider the following exogenous variable $X(\tau)$:

$$X(\tau) = \begin{cases} x(\tau) & 0 \leq \tau \leq T, \\ 0 & \tau > T, \end{cases}$$

where $x(\tau) > 0$, $x(\tau)$ is continuous on $[0, T]$, and $0 < T < t$. Then we get:

$$Y(t) = \int_0^t M(t,\tau)X(\tau)d\tau = \int_0^T M(t,\tau)x(\tau)d\tau. \quad \text{(A3)}$$

Eq. (A3) shows that even though for $t > T$ the exogenous variable $X(\tau)$ is equal to zero, the endogenous variable $Y(t)$ is nonzero (Tarasova and Tarasov 2018). This means that a memory of $X(\tau)$, $\tau \in [0, T]$, is stored in the process and present in $Y(t)$. This shows (*i*).

Now suppose for each t, the memory function $M(t, \tau)$ is continuous on $[0, T]$. According to mean value theorem for integrals (Apostol, 1979, p. 154), we can conclude that for each $t > T$ there exists $k_t \epsilon [0, T]$ such that:

$$Y(t) = M(t, k_t)X(k_t)T. \quad \text{(A4)}$$

Now suppose $\lim_{t\to\infty} M(t,\tau) = 0$ uniformly for $\tau\epsilon[0,T],\ t > T$. Because for each $t > T,\ k_t\epsilon[0,T]$, we have $\lim_{t\to\infty} M(t, k_t) = 0$. On the other hand, because $X(\tau)$ is continuous on $[0, T]$ and $k_t\epsilon[0, T]$ for any $t > T$, we can conclude that there exists an $L > 0$ such that $X(k_t) < L$ for any $t > T$. Now by Eq. (A4) we have:

$$|Y(t)| \leq |M(t, k_t)| * L * T,$$

which indicates that $\lim_{t\to\infty} Y(t) = 0$. In this case, we can clearly observe that the effect of memory fades over time which means that the economic process shown in Eq. (A1) forgets the $X(\tau)$, $\tau\epsilon[0, T]$, over time. This demonstrates (*ii*).

In the GDMR shown in Eq. (10) in which $M(t,\tau) = \frac{(t-\tau)^{\beta-1}}{\Gamma(\beta)}$, for $t > T$:

$$M(t, k_t) = \frac{(t-k_t)^{\beta-1}}{\Gamma(\beta)} < \frac{1}{(t-T)^{1-\beta}\Gamma(\beta)}, \quad \text{(A5)}$$

which means $\lim_{t \to \infty} M(t, k_t) = 0$ and we have a fading memory. It is clear that in the case of $\beta = 1$, $M(t, \tau) = 1$ and memory is not fading. By Eq. (A5) we can also observe that when β tends to zero the upper bound in Eq. (A5) tends to zero which means the economic process will have a lower memory of the past. This demonstrates (*iii*).

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Appendix B: Proofs

We start with a number of theorems and definitions before introducing a substitute operator. After introducing the new operator, by showing that the proposed operator, $I_{n,k}^{\beta}$, converges to I^{β} with respect to the operator norm, we demonstrate that $I_{n,k}^{\beta}$ has the same properties as those of I^{β}.

Theorem B1. On C[0, *T*], the Riemann-Liouville fractional integration operator has the semigroup property

$$I^{\delta}.I^{\lambda}(.) = I^{\delta+\lambda}(.). \tag{B1}$$

Here C[0, *T*] denotes the space of all continuous functions on the interval $[0, T]$ (Kilbas, Srivastave, and Trujillo. 2006).

Applying the semigroup property (B1) of fractional integral operator we have

$$I^{\beta}y(t) = I^{\beta}[y(0) + Iy'(0) + I^{2}y''(t)] \tag{B2}$$

$$= I^{\beta}y(0) + I^{1+\beta}y'(0) + I^{2+\beta}y''(t)$$

$$= \frac{y(0)t^{\beta}}{\Gamma(1+\beta)} + \frac{y'(0)t^{1+\beta}}{\Gamma(2+\beta)} + I^{2+\beta}y''(t).$$

We approximate operator I^{β} by applying the well-known n-point Gauss quadrature formula for integrals (DeVore and Scott 1984) and Spouge's approximate formula for the Gamma function (Spouge 1994). We first state Spouge's formula in the following theorem.

Theorem B2 (Spouge's approximation for the gamma function). For $x \in \mathbb{R}$, $x \geq 1$, the gamma function can be approximated as follows (Spouge 1994):

$$\Gamma(x) \cong (x-1+h)^{x-\frac{1}{2}}e^{-(x-1+h)}\sqrt{2\pi}\left[c_0 + \sum_{i=1}^{k}\frac{c_i(h)}{x-1+i}\right]. \tag{B3}$$

The parameter h is real, $k = \lceil h \rceil - 1$, $c_0 = 1$, and

$$c_i(h) = \frac{(-1)^{i-1}(-i+h)^{i-\frac{1}{2}}}{\sqrt{2\pi}(i-1)!}e^{-i+h}. \tag{B4}$$

There exists the error upper bound $\frac{\Gamma(x)}{(2\pi)^{h+\frac{1}{2}}\sqrt{h}}$ for approximate formula (B3), provided that $h \geq 3$.

Now set

$$G_k(x) = (x-1+h)^{x-\frac{1}{2}}e^{-(x-1+h)}\sqrt{2\pi}\left[c_0 + \sum_{i=1}^{k}\frac{c_i(h)}{x-1+i}\right] \tag{B5}$$

Applying the Gauss quadrature formula along with Eq. (A5) we approximate the operator $I^{2+\beta}$ by operator $I_{n,k}^{2+\beta}$ as follows:

$$I_{n,k}^{2+\beta}y''(t) := \frac{1}{G_k(2+\beta)}\frac{t}{2}\sum_{i=1}^{n} w_i\left(\frac{t}{2}(1-x_i)\right)^{1+\beta} y''\left(\frac{t}{2}(x_i+1)\right). \tag{B6}$$

In the above formula, $w_i$and x_i denote the quadrature nodes and weights (DeVore and Scott 1984). We consider the following operator, $I_{n,k}^{\beta}$, instead of the operator I^{β} given in Eq. (B2)

$$I_{n,k}^{\beta}y(t) := \frac{y(0)t^{\beta}}{G_k(1+\beta)} + \frac{y'(0)t^{1+\beta}}{G_k(2+\beta)} + I_{n,k}^{2+\beta}y''(t). \tag{B7}$$

We have substituted the operator I^{β} with the computationally implementable operator $I_{n,k}^{\beta}$. Now we need to demonstrate that $I_{n,k}^{\beta}$ maintains the desired characteristics of I^{β}. Specifically, we will show that, for large enough values of k and n, operator $I_{n,k}^{\beta}$ tends to the fractional integration operator I^{β}. We start with some definitions and theorems from functional analysis (Kreyszig 1978), which play a pivotal role in this argument.

The normed space $(C^3[0,T], \|.\|_3)$ is defined as follows:

$$C^3[0,T] = \{f(t)|\, f^{(3)}(t) \in C[0,T]\}, \tag{B8}$$

$$\|f\|_3 = \|f\|_\infty + \|f'\|_\infty + \|f''\|_\infty + \|f'''\|_\infty,$$

where C[0, *T*] denotes the space of continuous functions on the interval [0, *T*] equipped with the uniform norm,

$$\|f\|_\infty = \sup\{|f(t)| \mid t \in [0,T]\} \tag{B9}$$

Definition B1 (Bounded linear operator). Let $L: C^3[0,T] \to C[0,T]$ be a linear operator. The operator L is said to be bounded if there exists a real number c in such a way that for all $f \in C^4[0,T]$,

$$\|L(f)\|_\infty \leq c\|f\|_3.$$

Definition B2 (Operator Norm). Let L be a bounded linear operator as defined in definition A1. $\|L\|$ is called the norm of the operator L and is defined as

$$\|L\| = \inf\{c : \|L(f)\|_\infty \leq c\|f\|_3, for\ all\ f \in C^3[0,T]\}.$$

Theorem B3 presents an error upper bound for the Gauss quadrature formula.

Theorem B3. Let $E_n(f)$ denote the error in n-point Gaussian quadrature applied to function f on the interval $[0,t]$, $0 < t \leq T$. If $f' \in C[0,T]$, then

$$|E_n(f)| \leq \frac{3\pi t}{n}\int_0^t |f'(s)|\sqrt{1-(\frac{2}{t}s-1)^2 ds}.$$

Proof. (DeVore and Scott 1984)

Theorem B4. Consider $I_{n,k}^\beta$, $I^\beta: C^3[0,T] \to C[0,T]$, then as the values of $k\ and\ n$ increase, $\left\|I_{n,k}^\beta - I^\beta\right\|$ converges to zero.

Proof.

According to Theorem B2,

$$|\Gamma(\omega) - G_k(\omega)| < \frac{\Gamma(\omega)}{(2\pi)^{h+\frac{1}{2}}\sqrt{h}}, \tag{B10}$$

where $\omega \geq 1$, $k = \lceil h \rceil - 1$. According to inequality (B10), we can set k large enough so that,

$$|\Gamma(\omega) - G_k(\omega)| < \frac{1}{2}\Gamma(\omega).$$

Hence, we get $G_k(\omega) > \frac{1}{2}\Gamma(\omega)$; therefore,

$$\frac{1}{G_k(\omega)} < \frac{2}{\Gamma(\omega)}. \tag{B11}$$

Hence, from (B10) and (B11), we have

$$\left|\frac{1}{\Gamma(\omega)} - \frac{1}{G_k(\omega)}\right| \le \frac{2|\Gamma(\omega)-G_k(\omega)|}{\Gamma(\omega)^2} < \frac{2}{(2\pi)^{h+\frac{1}{2}}\sqrt{h}\,\Gamma(\omega)}. \tag{B12}$$

Let $f \in C^3[0,T]$. From Theorem B3, we get

$$\left|\int_0^t (t-s)^{1+\beta} f''(s)ds - \frac{t}{2}\sum_{i=1}^{n} w_i\left(\frac{t}{2}(1-x_i)\right)^{1+\beta} f''\left(\frac{t}{2}(x_i+1)\right)\right|$$

$$\le \frac{3\pi t}{n}\int_0^t [\left|(t-s)^{1+\beta}f^{(3)}(s) - (1+\beta)(t-s)^{\beta}f''(s)\right|\sqrt{1-(\frac{2}{t}s-1)^2}]ds \le$$

$$\frac{3\pi t}{n}\left(t^{1+\beta}\|f''\|_\infty + t^{2+\beta}\left\|f^{(3)}\right\|_\infty\right) \le \frac{3\pi}{n}\left(T^{2+\beta}+T^{3+\beta}\right)\|f\|_3. \tag{B13}$$

Referring to (B2) and (B7), applying (B11), (B12), and (B13) we get

$$\left|(I_{n,k}^{\beta} - I^{\beta})f(t)\right| =$$

$$\left|\frac{f(0)t^{\beta}}{\Gamma(1+\beta)} + \frac{f'(0)t^{1+\beta}}{\Gamma(2+\beta)}\right.$$

$$+\frac{1}{\Gamma(2+\beta)}\int_0^t (t-s)^{1+\beta}f''(s)ds - \frac{f(0)t^{\beta}}{G_k(1+\beta)} - \frac{f'(0)t^{1+\beta}}{G_k(2+\beta)}$$

$$\left.-\frac{1}{G_k(2+\beta)}\frac{t}{2}\sum_{i=1}^{n} w_i\left(\frac{t}{2}(1-x_i)\right)^{1+\beta} f''\left(\frac{t}{2}(x_i+1)\right)\right| \le$$

$$\left|\frac{1}{\Gamma(2+\beta)}\int_0^t (t-s)^{1+\beta}f''(s)ds\right.$$

$$-\frac{1}{G_k(2+\beta)}\int_0^t (t-s)^{1+\beta}f''(s)ds + \frac{1}{G_k(2+\beta)}\int_0^t (t-s)^{1+\beta}f''(s)ds$$

$$\left.-\frac{1}{G_k(2+\beta)}\frac{t}{2}\sum_{i=1}^{n} w_i\left(\frac{t}{2}(1-x_i)\right)^{1+\beta} f''\left(\frac{t}{2}(x_i+1)\right)\right| +$$

$$\left|f(0)T^{\beta}\right|\left|\frac{1}{\Gamma(1+\beta)}-\frac{1}{G_k(1+\beta)}\right|+\left|f'(0)T^{1+\beta}\right|\left|\frac{1}{\Gamma(2+\beta)}-\frac{1}{G_k(2+\beta)}\right|$$

$$\leq\left|\frac{1}{\Gamma(2+\beta)}-\frac{1}{G_k(2+\beta)}\right|\left(T^{2+\beta}\|f''\|_\infty+T^{1+\beta}\|f'\|_\infty\right)$$

$$+T^{\beta}\|f\|_\infty\left|\frac{1}{\Gamma(1+\beta)}-\frac{1}{G_k(1+\beta)}\right|+$$

$$\frac{1}{G_k(2+\beta)}\left|\int_0^t(t-s)^{1+\beta}f''(s)ds-\frac{t}{2}\sum_{i=1}^{n}w_i\left(\frac{t}{2}(1-x_i)\right)^{1+\beta}f''\left(\frac{t}{2}(x_i+1)\right)\right|$$

$$\leq\frac{2}{(2\pi)^{h+\frac{1}{2}}\sqrt{h}\Gamma(2+\beta)}\left(T^{2+\beta}\|f''\|_\infty+T^{1+\beta}\|f'\|_\infty\right)$$

$$+\frac{2}{(2\pi)^{h+\frac{1}{2}}\sqrt{h}\Gamma(1+\beta)}T^{\beta}\|f\|_\infty+\frac{3\pi}{nG_k(2+\beta)}\left(T^{2+\beta}+T^{3+\beta}\right)\|f\|_3\leq$$

$$\left(\frac{2}{(2\pi)^{h+\frac{1}{2}}\sqrt{h}\Gamma(2+\beta)}\left(T^{2+\beta}+T^{1+\beta}\right)+\frac{2}{(2\pi)^{h+\frac{1}{2}}\sqrt{h}\Gamma(1+\beta)}T^{\beta}+\frac{6\pi}{n\Gamma(2+\beta)}\left(T^{2+\beta}+T^{3+\beta}\right)\right)\|f\|_3.$$

Hence,

$$\left\|(I_{n,k}^{\beta}-I^{\beta})f\right\|_\infty\leq\left(\frac{2}{(2\pi)^{h+\frac{1}{2}}\sqrt{h}\Gamma(2+\beta)}\left(T^{2+\beta}+T^{1+\beta}\right)+\frac{2}{(2\pi)^{h+\frac{1}{2}}\sqrt{h}\Gamma(1+\beta)}T^{\beta}+\frac{6\pi}{n\Gamma(2+\beta)}\left(T^{2+\beta}+T^{3+\beta}\right)\right)\|f\|_3.$$

According to Definition A2, we observe that,

$$\left\|I_{n,k}^{\beta}-I^{\beta}\right\|\leq\left(\frac{2}{(2\pi)^{h+\frac{1}{2}}\sqrt{h}\Gamma(2+\beta)}\left(T^{2+\beta}+T^{1+\beta}\right)+\frac{2}{(2\pi)^{h+\frac{1}{2}}\sqrt{h}\Gamma(1+\beta)}T^{\beta}\right.$$

$$\left.+\frac{6\pi}{n\Gamma(2+\beta)}\left(T^{2+\beta}+T^{3+\beta}\right)\right).$$

Now it is easy to observe that as k and n increase, then $\left\| I_{n,k}^{\beta} - I^{\beta} \right\|$ converges to zero, thus proving the lemma. ■

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Appendix C: Additional Model Flexibility

The base GDMR can be further extended to allow for more model flexibility. For instance, under the periodic-repeat-purchases scenario ($\beta = 1$), sales grow at the same rate as does that of the corresponding cumulative adoptions, essentially assuming that adopters from the previous periods, on average, make one repeat purchase in each subsequent unit time period. However, there may exist scenarios in which the average frequency of repeat purchases is either higher or lower than 1. To incorporate such scenarios, we can introduce one more parameter, h, into the model:

$$S(t) = h^{\beta} * I^{\beta} y(t). \tag{C1}$$

Incorporating the new parameter h into Eq. (6), we have:

$$S(t) = h^{\beta} * I^{\beta} y(t) = \int_0^t h^{\beta} \frac{1}{\Gamma(\beta)} (t-\tau)^{(\beta-1)}\, y(\tau) d\tau, \tag{C2}$$

where $h^{\beta} \frac{1}{\Gamma(\beta)} (t-\tau)^{(\beta-1)}$ represents the extended memory function with higher flexibility than the one shown in Eq. (6) (i.e., $\left[\frac{1}{\Gamma(\beta)} (t-\tau)^{(\beta-1)}\right]$).

The new parameter h in Eq. (C1) or (C2) has the effect of "raising" ($h > 1$) or "lowering" ($0 < h < 1$) the sales curve along the vertical dimension. For instance, when $\beta = 1$, for $h > 1$, adopters on average make more than one repeat purchase in each time-period, and for $0 < h < 1$, adopters on average make less than one repeat purchase in each time-period. We would like to point out, however, that the additional parameter should not be included unless it leads to clear improvement in model fitting or prediction.

Appendix D: Model Evaluation Based on Simulation

One of the most important characteristics of the *Generalized Diffusion Model with Repeat Purchases* (GDMR) is that it can be directly fit to aggregate sales data without knowing the percentages of adoptions (i.e., first-time purchases) and repeat purchases in sales. This is a very critical advantage because the vast majority of sales datasets available to researchers and analysts, including those used in the prior literature to test diffusion models, are aggregate sales data — it is much easier for firms to keep track of sales than adoptions and repeat purchases. Once the model parameter values are obtained from sales data based on the GDMR, it is straightforward to estimate the amount of adoptions and repeat purchases. To empirically evaluate how reliably the GDMR separates adoptions from repeat purchases, however, we need to have both sales data and adoption data. Unfortunately, in practice, most businesses are primarily interested in total sales, and have limited insensitive or resources to collect separate data on adoptions and repeat purchases. As a result, sales data that includes the breakdown of adoptions and repeat purchase are very difficult to obtain. For this reason, we conduct simulations to generate sales and adoption data, which are subsequently used to evaluate the GDMR's performance, particularly in recovering adoption trend from sales data, and in linking the amount of repeat purchases to the coefficient of repeat purchases (β).

Simulation Procedure

Consistent with the motivation of the GDMR in the paper, in our simulation, product sales are generated through adoptions and repeat purchases (including both replacement and multi-unit ownership purchases). In addition, to be consistent with reality (newer and better products will eventually emerge and buyers will jump ship) and trends observed from real data (even with repeat purchases, sales drop after some period of time for all products), we also simulate buyers' departures from a focal product. We next describe each of the components in detail.

Adoptions. We simulate a population of 1000 potential adopters (n=1000). We generate the time to adoption for each adopter by calling a random number generator based on the distribution suggested by the Bass Model (Bass 1969), because it is the most widely applied diffusion model and hence the golden standard. We repeat the procedure 1000 times to generate the time of adoption for all simulated adopters.

Multi-unit Ownership Purchases. We simulate only second-unit purchases because we believe that they are sufficient to capture the essence of the multi-unit ownership phenomenon, and for many products, a small percentage of users own more than two units of the product. We select a random subset of adopters who proceed to purchase a second unit. Following Steffens (2003), we treat these purchases as adoptions, *second adoptions* to be specific, and assume that the time-to-second-adoptions also follows the distribution suggested by the Bass Model.

Replacement Purchases. Following Kamakura and Balasubramanian (1987), we assume that product replacements follow an h distribution. In the two-unit-ownership cases, we assume that the replacement of the first unit occurs independently of the replacement of the second unit. Furthermore, the time to replacement for each adoption is independently and identically distributed. We generate the time-to-replacements for each adoption based on the h distribution. A replacement can be replaced again as time goes by. The time of occurrence of the n^{th} replacement is the summation of the times-to-replacement for the first n replacements.

Departures. Departures take place as a result of users switching to an alternative product. Therefore, we simulate departures using a Bass diffusion process. We randomly select a subset of users who depart. The essence of a departure is that replacements will no longer be generated after the departure. In the case of two-unit-ownerships, the departures of the first and second adoptions are independent, implying that a user who owns two units may decide to stop replacing one unit while keep replacing the other. In our simulation, departures start sometime after the focal product's first adoption, implying that the competing product does not emerge immediately after the launch of the focal product.

Parameter Value Selection

Because of the number of parameters needed for our simulation, a random combination of parameter values could lead to unrealistic scenarios. To construct realistic simulation scenarios, we first simulate an *anchor scenario* based on the available DVD players dataset that records both product adoptions and sales. We label this scenario as *Anchor I*. We subsequently generate two more anchor scenarios by shrinking and extending the adoption time-frame of the Anchor I, to accommodate products with faster and slower adoption processes than DVD players. They are named *Anchor II* (shorter time frame), and *Anchor III* (longer time frame), respectively.

To simulate a wider variety of product categories and market conditions, for each of the three anchor scenarios, we alter its sales components to generate more scenarios. Specifically, we change the following parameters one at a time: (1) the percentage of adopters who adopt a second unit, (2) the average service life of the product, (3) the percentage of adopters who depart, (4) the departure start time, and (5) the ratios of coefficient of innovation to coefficient of imitation for first adoptions, second adoptions, and departures.

We next describe how the Anchor I scenario is generated based on the DVD players dataset. We use the coefficient of innovation (p) and the coefficient of imitation (q) derived from fitting the Bass Model to the DVD players adoption data. We assume that 50% of the adopters adopted a second unit, because records show that more than 50% percent of the US households owned more than three TV sets (News Wire 2009), and if at least two of these TV sets were equipped with a DVD player, we can conclude that at least 50% of the DVD users adopted a second unit. In addition, because second-unit adopters already had their own experience with the product and were less influenced by other adopters, we assume that second adoptions are more likely to be driven by independent decision making. Therefore, we use a higher coefficient of innovation for second adoptions than that for first adoptions. Regarding replacement purchases, based on reported records (Consumer Technology Association 2014), we assume that the simulated product's average service life is 5 years. The h distribution we implement for product replacement has a shape parameter that reflects the relationship between maximum product service life and average product service life. Following Kamakura and Balasubramanian (1987), we set $h = 1.75$, meaning that the maximum service life is three times as long as the average service life. Our empirical tests show that the results are not highly sensitive to the value of this shape parameter. In the DVD player case, the emergence of the alternative (i.e., Blu-Ray) and subsequently the departure of users started nine years after the release of the DVD technology and when the product adoption was 90% complete. We set the departure start time for our first anchor scenario in the same manner. Because Blu-Ray players and DVD players are similar products, we further assume that the coefficient of innovation and the coefficient of imitation for the departure process are similar to those of the focal product's adoption process. The last parameter we need to determine is the percentage of departing adoptions. We select this percentage in a way such that fitting the GDMR to the simulated sales data results in a repeat purchases parameter value close to what we get from fitting the GDMR to the real DVD players dataset — we select the percentage of departures to be 70%.

As explained earlier, Anchor II and Anchor III scenarios are generated by shrinking and extending the adoption time-frame of the Anchor I scenario generated using the DVD players dataset. The shrinkage and extension of the adoption time frame are done by altering the coefficient of imitation. Specifically, in the DVD players case, 99% of adoptions were completed in 13 years. In the Anchor II scenario, 99% of adoptions are completed in 10 years. In the Anchor III scenario, 99% of adopters are completed in 16 years. Similar to the Anchor I scenario, in the Anchor II and Anchor III scenarios, we assume that the coefficient of innovation for second adoption is larger than that for the first adoption. For the Anchor I scenario, departures start to occur when first adoptions are 90% complete. The departure times for the other two anchor scenarios are set in a similar manner.

As explained earlier, for each of the three anchor scenarios developed, we change the values of five types of parameters one at a time. (1) the percentage of second adoptions, (2) the replacement cycles, (3) the departure start time, (4) the percentage of departures, and (5) the ratios of coefficient of innovation to coefficient of imitation in the first adoptions, second adoptions, and departures. We next provide additional details regarding how the values of these parameters are set.

(1) *Percentage of second adoptions*. In addition to the default 50%, we also consider 30% and 10% of second adoptions.

(2) *Average service life of the product*. In addition to the default average service life of 5 years, we also consider 4, 7, and 9 years of average service life.

(3) *Departure start time*. In the three anchor scenarios, the emergence of alternative products and departures of the users for the alternatives start when the product adoption is 90% complete. For variation scenarios, we consider earlier user departure start times for the three anchor scenarios.

For Anchor I, adoption peaks at around t=7.7, and we let departures start at t=7. For Anchor II, adoptions peak at t=6.2, and we let departures start at t=6. For Anchor III, adoptions peak at t=9.3, and we consider two earlier departure start times at t=9 and t=10, respectively.

4) *Percentage of adopters who depart*. For the three anchor scenarios, in addition to the default 70%, we also consider 50% and 90% of departures.

(5) *Ratio of coefficient of innovation to coefficient of imitation.* For the three anchor scenarios, we also consider scenarios with a high coefficient of innovation for first adoptions, up to twenty times as large as the coefficient of innovation estimated from the DVD players adoption data. Altogether, the procedures descried above generate 31 scenarios. We report the parameter values for these 31 scenarios in Tables A1 to A3.

Table A1. Parameter Values for Anchor I and Corresponding Variation Scenarios

	Adoption		Repeat Purchases					Departure			
			Second Adoption			Replace ment					
	p_1	q_1	p_2	q_2	m_2	l	h	p_d	q_d	m_d	t*
Anchor I– 1	0.003	0.77	1.2	1	0.5	5	1.75	0.003	0.77	0.7	9
Changing Average Service Life											
Anchor I – 2	0.003	0.77	1.2	1	0.5	**4**	1.75	0.003	0.77	0.7	9
Anchor I – 3	0.003	0.77	1.2	1	0.5	**7**	1.75	0.003	0.77	0.7	9
Anchor I – 4	0.003	0.77	1.2	1	0.5	**9**	1.75	0.003	0.77	0.7	9
Changing Second Adoption Percent											
Anchor I – 5	0.003	0.77	1.2	1	**0.3**	5	1.75	0.003	0.77	0.7	9
Anchor I – 6	0.003	0.77	1.2	1	**0.1**	5	1.75	0.003	0.77	0.7	9
Changing the Departure Start Time											
Anchor I – 7	0.003	0.77	1.2	1	0.5	5	1.75	0.003	0.77	0.7	**7**
Changing Departure Percentage											
Anchor I – 8	0.003	0.77	1.2	1	0.5	5	1.75	0.003	0.77	**0.5**	9
Anchor I – 9	0.003	0.77	1.2	1	0.5	5	1.75	0.003	0.77	**0.9**	9
Changing Adoption Parameters											

Anchor I – 10	**0.06**	**0.5**	**1.5**	**1.3**	0.5	5	1.75	**0.06**	**0.5**	0.7	**6.5**

Table A2. Parameter Values for Anchor II and Corresponding Variation Scenarios

	Adoption		Repeat Purchases					Departure			
			Second Adoption			Replace ment					
	p_1	q_1	p_2	q_2	m_2	l	h	p_d	q_d	m_d	t^*
Anchor II – 1	0.003	1	1.4	1.2	0.5	5	1.75	0.003	1	0.7	7.2
Change Average service Life											
Anchor II – 2	0.003	1	1.4	1.2	0.5	**4**	1.75	0.003	0.77	0.7	7.2
Anchor II – 3	0.003	1	1.4	1.2	0.5	**7**	1.75	0.003	0.77	0.7	7.2
Anchor II – 4	0.003	1	1.4	1.2	0.5	**9**	1.75	0.003	0.77	0.7	7.2
Changing Second Adoption Percent											
Anchor II – 5	0.003	1	1.4	1.2	**0.3**	5	1.75	0.003	0.77	0.7	7.2
Anchor II – 6	0.003	1	1.4	1.2	**0.1**	5	1.75	0.003	0.77	0.7	7.2
Changing the Departure Start Time											
Anchor II – 7	0.003	1	1.4	1.2	0.5	5	1.75	0.003	0.77	0.7	**6**
Changing Departure percentage											
Anchor II – 8	0.003	1	1.4	1.2	0.5	5	1.75	0.003	0.77	**0.5**	7.2
Anchor II – 9	0.003	1	1.4	1.2	0.5	5	1.75	0.003	0.77	**0.9**	7.2
Changing Adoption Parameters											
Anchor II – 10	**0.06**	**0.65**	**1.6**	**1.4**	0.5	5	1.75	**0.06**	**0.65**	0.7	**5.5**

Table A3. Parameter Values for Anchor III and Corresponding Variation Scenarios

	Adoption		Repeat Purchases					Departure			
			Second Adoption			Replace ment					
	p_1	q_1	p_2	q_2	m_2	l	h	p_d	q_d	m_d	t^*
Anchor III– 1	0.003	0.6	1	0.8	0.5	5	1.75	0.003	0.6	0.7	11.1
Change Average service Life											
Anchor III – 2	0.003	0.6	1	0.8	0.5	**4**	1.75	0.003	0.6	0.7	11.1
Anchor III – 3	0.003	0.6	1	0.8	0.5	**7**	1.75	0.003	0.6	0.7	11.1

Anchor III – 4	0.003	0.6	1	0.8	0.5	**9**	1.75	0.003	0.6	0.7	11.1
Changing Second Adoption Percent											
Anchor III – 5	0.003	0.6	1	0.8	**0.3**	5	1.75	0.003	0.6	0.7	11.1
Anchor III – 6	0.003	0.6	1	0.8	**0.1**	5	1.75	0.003	0.6	0.7	11.1
Changing the Departure Start Time											
Anchor III – 7	0.003	0.6	1	0.8	0.5	5	1.75	0.003	0.6	0.7	**10**
Anchor III – 8	0.003	0.6	1	0.8	0.5	5	1.75	0.003	0.6	0.7	**9**
Changing Departure percentage											
Anchor III – 9	0.003	0.6	1	0.8	0.5	5	1.75	0.003	0.6	**0.5**	11.1
Anchor III– 10	0.003	0.6	1	0.8	0.5	5	1.75	0.003	0.6	**0.9**	11.1
Changing Adoption Parameters											
Anchor III – 11	**0.06**	**0.4**	**1.4**	**1.2**	0.5	5	1.75	**0.06**	**0.4**	0.7	**7.5**

Results of Performance Evaluation

To evaluate model performance, we first fit the GDMR to the simulated aggregate sales data, and then use the resulting adoption parameters (p, q, and m) to estimate the adoption trends. In addition, we evaluate the model's three-years-ahead sales forecasting performance by withholding three last sales data points, fitting the GDMR to the remaining sales data points, and finally using the three withheld sales data points to measure the forecasting performance of the GDMR.

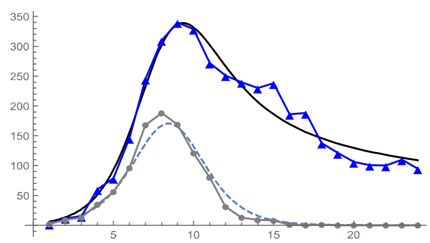
The parameter estimates, model fit (both sales and adoptions) performance in terms of mean absolute percentage error (MAPE), and forecasting performance in terms of MAPE, for the three anchor scenarios and their variations are summarized in Tables A4 to Table A6. From these results, we can see that all parameter estimates are significant for all scenarios. Most of the model fit and forecasting measured in MAPE are also quite reasonable.

To better asses the GDMR's performances, we create Figures A1 to A3 to show the GDMR's fit to sales and adoptions. With a few exceptions, the estimates by the GDMR are also close to the simulated sales and adoption data. This demonstrates that the GDMR can retrieve adoption trend from sales trend with high accuracy. We would like to note, however, that the adoption trends estimated by the GDMR for the first and second anchor scenarios and their corresponding variations are more accurate than those estimated for the third anchor scenario and its variations. This suggests that the GDMR performs better in cases where adoption takes place in a relatively shorter time frame, which is likely to be true in today's fast-paced technology product markets. Collectively, these empirical results give us confidence that the GDMR can reliably estimate sales and recover adoption trends from sales data for a wide range of product categories, especially those with average or relatively faster speed of market penetration.

Table A4. Model Estimation and Forecasting Performance for Anchor I and Variation Scenarios

	Parameters Estimates				Model Fit (MAPE)		Three-Years-Ahead Sales Forecast (MAPE)
	β	p	q	m	Adoption	Sales	
Anchor I– 1	0.44 (***)	0.003 (***)	0.67 (***)	1018.9 (***)	42.0	22.7	20.1
Changing Average Service Life							
Anchor I – 2	0.47 (***)	0.002 (***)	0.75 (***)	900.8 (***)	24.0	14.2	13.3
Anchor I – 3	0.4 (***)	0.002 (***)	0.76 (***)	904.7 (***)	21.8	12.33	9.6
Anchor I – 4	0.34 (***)	0.001 (***)	0.84 (***)	899.4 (***)	18.5	10.2	11.36
Changing Second Adoption Percent							

Anchor I – 5	0.4 (***)	0.003 (***)	0.64 (***)	1002 (***)	52.6	23.1	18.4
Anchor I – 6	0.62 (***)	0.004 (***)	0.6 (***)	918 (***)	64.1	33.2	37.1
Changing the Departure Start Time							
Anchor I – 7	0.41 (***)	0.002 (***)	0.71 (***)	1040.2 (***)	39.3	15.1	8.0
Changing Departure Percentage							
Anchor I – 8	0.59 (***)	0.001 (*)	0.92 (***)	686 (***)	39.3	11.4	4.5
Anchor I – 9	0.67 (***)	0.004 (*)	0.61 (***)	1283.0 (***)	178.9	32.3	44.9
Changing Adoption Parameters							
Anchor I – 10	0.42 (***)	0.044 (***)	0.49 (***)	993.6 (***)	8.0	6.9	17.9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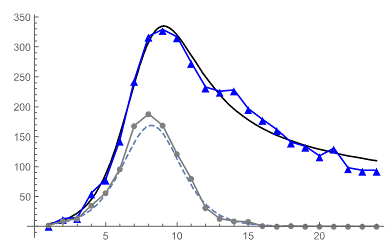
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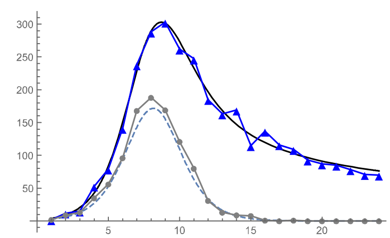
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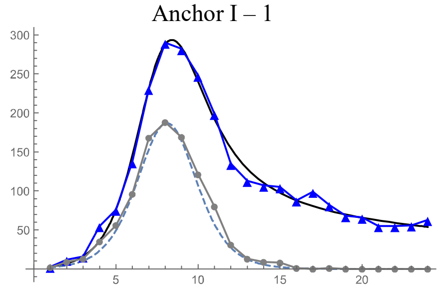
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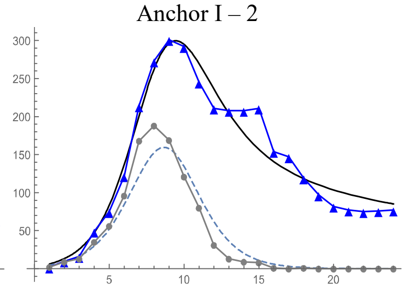
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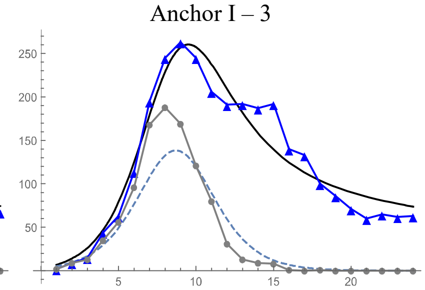
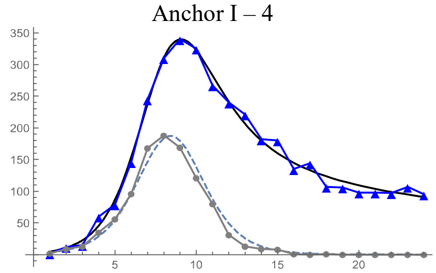

Anchor I – 6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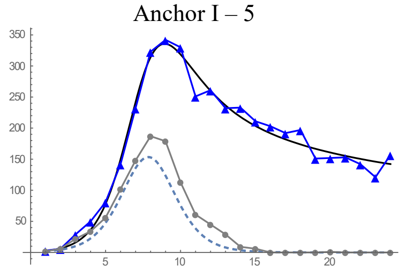
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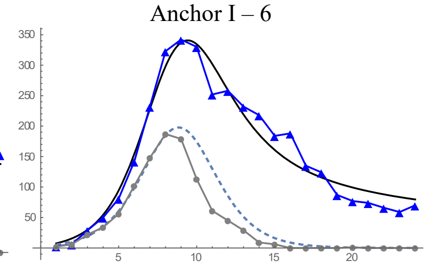
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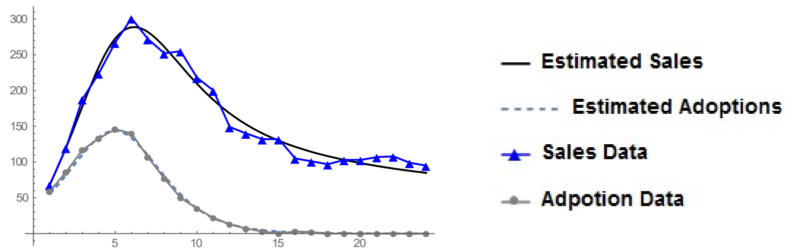


Anchor I – 10

Figure A1. GDMR's Fit on Sales and Adoptions for Anchor I and Variation Scenarios

Table A5. Model Estimation and Forecasting Performance for Anchor II and Variation Scenarios

	Parameters Estimates				Model Fit (MAPE)		Three-Years-Ahead GDMR Sales Forecast (MAPE)
	β	*p*	*q*	*m*	Adoption	Sales	
Anchor II– 1	0.44 (***)	0.002 (*)	0.93 (***)	961 (***)	16.0	10.6	4.9
Changing Average Service Life							
Anchor II – 2	0.44 (***)	0.002 (***)	1.0 (***)	933.9 (***)	15.7	9.3	16.6
Anchor II – 3	0.37 (***)	0.002 (***)	1.0 (***)	938.0 (***)	15.6	8.8	16.1
Anchor II – 4	0.28 (***)	0.002 (***)	1.0 (***)	1036.1 (***)	18.6	12.5	25.7
Changing Second Adoption Percent							
Anchor II – 5	0.38 (***)	0.003 (***)	0.86 (***)	945.8 (***)	29.2	16.1	7.0
Anchor II – 6	0.36 (***)	0.005 (**)	0.76 (***)	887.0 (***)	19.8	58.0	6.6
Changing the Departure Start Time							
Anchor II – 7	0.39 (***)	0.002 (***)	1 (***)	10.25.0 (***)	14.3	12.2	24.8
Changing Departure Percentage							
Anchor II – 8	0.51 (***)	0.002 (*)	1.0 (***)	840.6 (***)	20.1	11.3	18.5
Anchor II – 9	0.33 (***)	0.002 (*)	0.96 (***)	1162.1 (***)	40.9	17.7	14.9

Changing Adoption Parameters							
Anchor II – 10	0.44 (***)	0.05 (***)	0.67 (***)	872.7 (***)	29.5	10.1	23.6

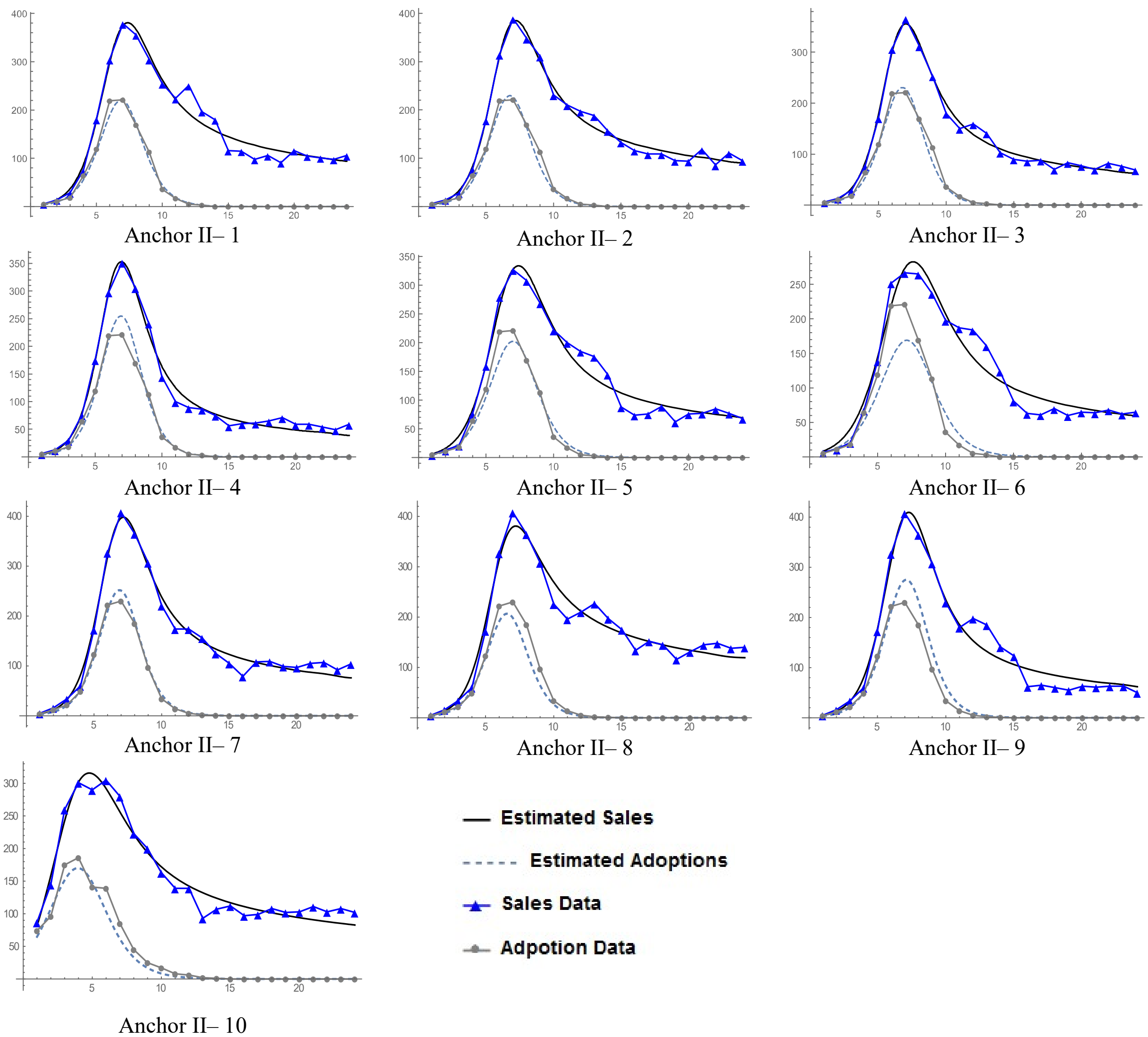
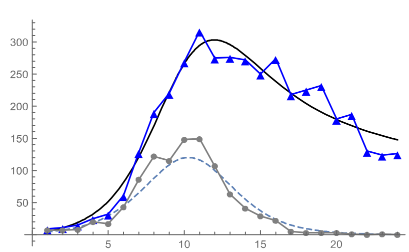


Figure A2. GDMR's Fit on Sales and Adoptions for Anchor II and Variation Scenarios

Table A6. Model Estimation and Forecasting Performance for Anchor III and Variation Scenarios

	Parameters Estimated by Fitting the GDMR on the Simulated Sales Trend	GDMR Sales Fit (MAPE)	Three-Years-Ahead GDMR Sales Forecast (MAPE)

	β	p	q	m	Adoption	Sales	
Anchor III– 1	0.5 (***)	0.004 (**)	0.5 (***)	944.6 (***)	43.4	13.8	46.7
Changing Average Service Life							
Anchor III – 2	0.53 (***)	0.003 (**)	0.57 (***)	828.6 (***)	21.3	11.28	27.0
Anchor III – 3	0.5 (***)	0.002 (**)	0.64 (***)	773.2 (***)	29.1	14.4	30.4
Anchor III – 4	0.43 (***)	0.001 (**)	0.67 (***)	730.3 (***)	34.1	16.5	27.5
Changing Second Adoption Percent							
Anchor III – 5	0.44 (***)	0.004 (**)	0.46 (***)	997.8 (***)	70.1	19.1	62.7
Anchor III – 6	0.60 (***)	0.005 (**)	0.42 (***)	96.9 (***)	99.5	21.6	67.6
Changing the Departure Start Time							
Anchor III – 7	0.45 (***)	0.003 (**)	0.50 (***)	1048.7 (***)	60.0	13.3	56.3
Anchor III – 8	0.42 (***)	0.003 (***)	0.05 (***)	1113.0 (***)	67.0	12.3	32.2
Changing Departure Percentage							
Anchor III – 9	0.63 (***)	0.003 (**)	0.58 (***)	700.7 (***)	26.0	12.7	27.9
Anchor III – 10	0.36 (***)	0.004 (**)	0.44 (***)	1368.3 (***)	154.0	21.1	74.8
Changing Adoption Parameters							
Anchor III – 11	0.41 (***)	0.04 (***)	0.37 (***)	1070.3 (***)	48.2	8.9	10.1



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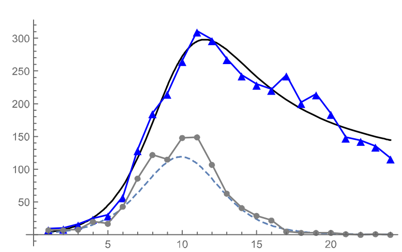
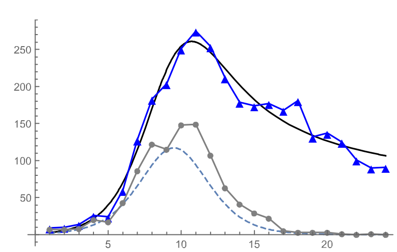


Anchor III – 2



Anchor III – 3

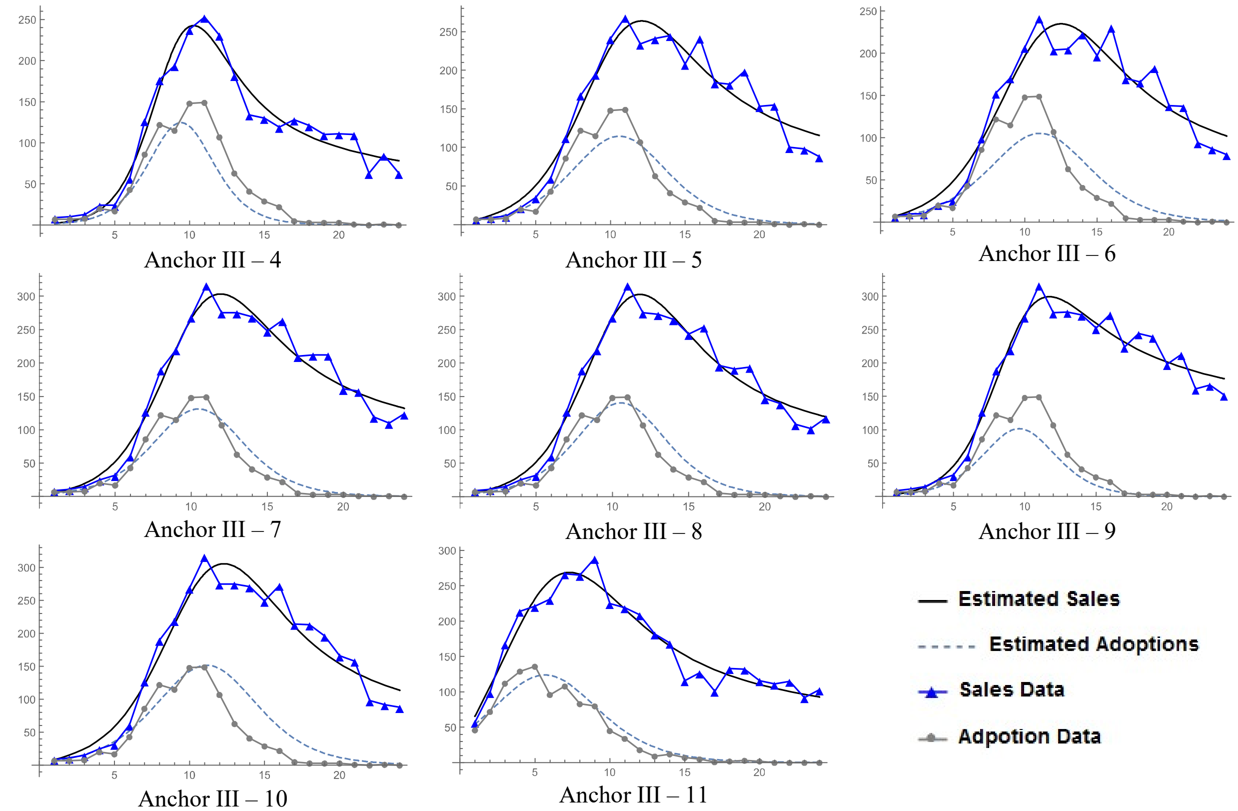


Figure A3. GDMR's Fit on Sales and Adoptions for Anchor III and Variation Scenarios

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